TIME DEPENDENT GENERATOR COORDINATE METHOD AND THE N-N INTERACTION *

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ABSTRACT – An attempt is made to describe the nucleon – nucleon interaction in a non – relativistic situation. Explicit expressions are obtained using a non – perturbative method appropriate to the wave packet which describes moving particles.

1 — INTRODUCTION. GENERAL CONSIDERATIONS

The question of the nature of the force between two nucleons has occupied a central place in physics.

Although Yukawa's theory, explaining the nuclear interaction as a result of the exchange of mesons between nucleons, dates back to 1935, the most important calculations of the nucleon-nucleon interaction in terms of π meson exchange was first carried out about 1950.

In this paper we wish to investigate the interaction energy between two nucleons, in a state of relative motion, from a non-perturbative viewpoint.

We consider the system nucleon – meson cloud in a very simplified version, that is, we restrict our discussion to scalar mesons exchanged between scalar nucleons.

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In this way the Hamiltonian may be written:

$$H = \sum_{p} (p^2/2m)\, C_p^+\, C_p + \sum_{k} \omega_k\, A_k^+\, A_k + G \sum_{p,k} (2\omega_k V)^{-1/2}\, [A_k^+\, C_p^+\, C_{p+k} + C_{p+k}^+\, C_p\, A_k] \tag{1}$$

where C_p^+ (C_p) creates (annihilates) a nucleon with momentum **p** and A_k^+ (A_k) creates (annihilates) a meson with momentum **k**.

In equation (1), m denotes the nucleon mass and $\omega_k = (k^2 + \mu^2)^{1/2}$, where μ is the meson mass. G is the coupling constant between nucleons and mesons and V refers to the normalization volume.

The operators $C_p^+ (C_p)$ and $A_k^+ (A_k)$ obey the usual commutation relations for fermions and bosons, respectively.

First, we consider the problem of one nucleon state and we calculate the expression of the energy of this system. After the interpretation of these results we follow the same procedure for a system of two interacting nucleons.

2 — ONE NUCLEON STATES

We construct the wave function for the system, as usually [1] [2], using the variational method, proceeding in analogy with the Hartree-Fock theory. To do this, we first consider the state $|\zeta>$ such that

$$A_k\, |\zeta> = \zeta_k\, |\zeta> \tag{2}$$

$$C_p\, |\zeta> = 0 \tag{3}$$

where ζ_k is a parameter. $|\zeta>$ is a coherent state given by

$$|\zeta> = \mathcal{N} \exp\, (\sum_{k} \zeta_k\, A_k^+)\; |0> \tag{4}$$

where $\mathcal{N}$ is a normalization constant and $|0>$ is the absolute vacuum,

$$A_k\, |0> = C_p\, |0> = 0 \tag{5}$$

A one-nucleon state may be obtained by the following construction

$$|\psi, \zeta> = \sum_{\mathbf{p}} \psi_{\mathbf{p}} \, C^{+}_{\mathbf{p}} \, |\zeta> \tag{6}$$

The ket $|\psi, \zeta>$ is normalized to unity provided the amplitudes $\psi_{\mathbf{p}}$ satisfy the relation

$$\sum_{\mathbf{p}} \psi^{*}_{\mathbf{p}} \, \psi_{\mathbf{p}} = 1 \tag{7}$$

The quantities $\psi_{\mathbf{p}}$ and $\zeta_{\mathbf{k}}$ are variational parameters obtained by minimization of the expectation value

$$<\psi, \zeta \,|\, H \,|\, \psi, \zeta> / <\psi, \zeta \,|\, \psi, \zeta> \tag{8}$$

We introduce the following simplifying assumption for $\psi_{\mathbf{p}}$

$$\psi_{\mathbf{p}} = [\,(4\pi\lambda)^{3/2} / V\,]^{1/2} \exp\,(-\tfrac{1}{2}\lambda\, p^2) \tag{9}$$

where λ is a parameter.

If we consider now the system placed on a point of space with positional vector $\mathbf{r}$, the function given by eq. (6) may be written

$$|\psi, \zeta> = \sum_{\mathbf{p}} \psi_{\mathbf{p}} \, e^{-i\,\mathbf{p}\,.\,\mathbf{r}} \, C^{+}_{\mathbf{p}} \, \exp\,(\sum_{\mathbf{k}} \zeta_{\mathbf{k}} \, e^{-i\,\mathbf{k}\,.\,\mathbf{r}} \, A^{+}_{\mathbf{k}}) \,|\, 0> \tag{10}$$

The ket $|\psi, \zeta>$, with the structure given by eq. (10), represents a static description of the system nucleon-meson [3]. The introduction of the velocity in that wave function may be done by writting

$$|\psi, \zeta> = \sum_{\mathbf{p}} \psi\,(\mathbf{p} - \mathbf{p}_0) \, e^{-i\,\mathbf{p}\,.\,\mathbf{r}} \, C^{+}_{\mathbf{p}} \, \exp\,(\sum_{\mathbf{k}} \zeta_{\mathbf{k}} \, e^{-i\,\mathbf{k}\,.\,\mathbf{r}} \, A^{+}_{\mathbf{k}}) \,|\, 0> \tag{11}$$

where $\mathbf{p}_0$ is a new variational parameter.

In order to obtain the expression of the parameter $\zeta_{\mathbf{k}}$ the minimization of eq. (8) should be performed by constraining the momentum operator $\mathbf{P}$ which may be written

$$\mathbf{P} = \sum_{\mathbf{p}} \mathbf{p} \, C^{+}_{\mathbf{p}} \, C_{\mathbf{p}} + \sum_{\mathbf{k}} \mathbf{k} \, A^{+}_{\mathbf{k}} \, A_{\mathbf{k}} \tag{12}$$

The expectation value of $(H - \mathbf{b}\cdot\mathbf{P})$ where $\mathbf{b}$ is a Lagrange multiplier, leads to

$$< \psi, \zeta \,|\, (H - \mathbf{b}\cdot\mathbf{P}) \,|\, \psi, \zeta > = (1/2\,m)\,(p_0^2 + 3/(2\,\lambda)) + \sum_{\mathbf{k}} \omega_k\, \zeta^*_k\, \zeta_k + G \sum_{\mathbf{k}} (2\,\omega_k\, V)^{-1/2}\; (\zeta^*_k + \zeta_k) \quad (13)$$
$$\exp\,(-\tfrac{1}{4}\lambda\, k^2) - \mathbf{b}\cdot\mathbf{p}_0 - \mathbf{b}\cdot\sum_{\mathbf{k}} \mathbf{k}\; \zeta^*_{\mathbf{k}}\, \zeta_{\mathbf{k}}$$

To obtain eq. (13) we have used the relations

$$\sum_{\mathbf{p}} \psi^*_{\mathbf{p}}\; \psi_{\mathbf{p}} = 1$$
$$\sum_{\mathbf{p}} \psi^*_{\mathbf{p}}\; p^2\; \psi_{\mathbf{p}} = p_0^2 + 3/(2\,\lambda)$$
$$\sum_{\mathbf{p}} \psi_{\mathbf{p}+\mathbf{k}}\; \psi_{\mathbf{p}} = \exp\,(-\tfrac{1}{4}\lambda\, k^2)$$

Variation of eq. (13) with respect to $\zeta_{\mathbf{k}}$ yields

$$\zeta_{\mathbf{k}} = -G\,(2\,\omega_k\, V)^{-1/2}\,(\omega_k - \mathbf{b}\cdot\mathbf{k})^{-1}\,\exp\,(-\tfrac{1}{4}\lambda\, k^2) \quad (14)$$

Further variation with respect to $\mathbf{p}_0$ yields $\mathbf{p}_0 = \mathbf{b}\; m$ which suggests that one can identify the quantity $\mathbf{b}$ as the veloctity of the system.

The energy expectation value of the nucleon-meson system is then

$$\mathscr{H}_1 = <\psi, \zeta \,|\, H \,|\, \psi, \zeta> / <\psi, \zeta \,|\, \psi, \zeta> = 3/(4\,m\,\lambda) + (m/2)\,\mathbf{b}\cdot\mathbf{b}$$
$$-\,G \sum_{\mathbf{k}} [\,1/(2\,\omega_k^2) - (\mathbf{b}\cdot\mathbf{k})^2/(2\,\omega_k^4)\,]\; \exp\,(-\tfrac{1}{2}\lambda\, k^2) \quad (15)$$
$$\simeq 3/(4\,m\,\lambda) - G^2/(4\,\pi\sqrt{2\,\pi\,\lambda}) + 1/2\;\mathbf{b}\cdot\mathbf{b}\;[\,m + G^2/(6\,\pi\sqrt{2\,\pi\,\lambda})\,]$$

where we have used the expansion of the expression of ζ_k (eq. 14) up to second order in powers of $\mathbf{b}$ and performed the summation over $\mathbf{k}$.

Comparing this expression with the expectation value of the momentum operator which is

$$< \psi, \zeta \,|\, \mathbf{P} \,|\, \psi, \zeta > = \mathbf{b}\;[\,m + G^2/(6\,\pi\sqrt{2\,\pi\,\lambda})\,] \quad (16)$$

we may notice that:

i) the interpretation already given for the parameter **b** is confirmed;
ii) it is possible to define the nucleon mass as

$$\mathscr{M} = \mathrm{m} + \mathrm{G}^2 / (6\pi\sqrt{2\pi\lambda}) \tag{17}$$

When we write the Lagangian of this system

$$\mathscr{L} = \dot{\mathbf{r}} \cdot \mathbf{b} [\mathrm{m} + \mathrm{G}^2 / (6\pi\sqrt{2\pi\lambda})] - \mathscr{H}_1 \tag{18}$$

we can also define the momentum canonically conjugate to **r** as

$$\mathbf{p} = \mathscr{M} \; \mathbf{b} \tag{19}$$

3 — TWO NUCLEON STATES: THE NUCLEAR INTERACTION

In analogy with the procedure followed in the previous section to construct the one-nucleon states, we may now write the wave function to describe two nucleons with opposite velocities **b** and $-$ **b** and localized at points a distance $|\mathbf{r}|$ apart

$$\psi_{1,2}(\mathbf{r}, \mathbf{b}) = \sum_{\mathbf{p}, \mathbf{p}'} \psi(\mathbf{p} - \mathbf{b}\,\mathrm{m})\, e^{-i\,\mathbf{p}\,.\,\mathbf{r}/2}\, \mathrm{C}^+_{\mathbf{p}} \exp\big(\sum_{\mathbf{k}} \zeta_{\mathbf{k},\mathbf{b}}\, e^{-i\,\mathbf{k}\,.\,\mathbf{r}/2}\, \mathrm{A}^+_{\mathbf{k}}\big)$$
$$\psi(\mathbf{p}' + \mathbf{b}\,\mathrm{m})\, e^{i\,\mathbf{p}'\,.\,\mathbf{r}/2}\, \mathrm{C}^+_{\mathbf{p}'} \exp\big(\sum_{\mathbf{k}'} \zeta_{\mathbf{k}',\mathbf{b}}\, e^{i\,\mathbf{k}'\,.\,\mathbf{r}/2}\, \mathrm{A}^+_{\mathbf{k}'}\big) \,|\,0> \tag{20}$$

With the Hamiltonian written in eq. (1) and eq. (20) we can write

$$\mathscr{H}_{12} = <\psi_{1,2}(\mathbf{r},\mathbf{b})\,|\,\mathrm{H}\,|\,\psi_{1,2}(\mathbf{r},\mathbf{b})> / <\psi_{1,2}(\mathbf{r},\mathbf{b})\,|\,\psi_{1,2}(\mathbf{r},\mathbf{b})>$$
$$= [3/(2\,\mathrm{m}\,\lambda) + \mathrm{b}^2\,\mathrm{m}] + \sum_{\mathbf{k}} \omega\, [\zeta^2_{\mathbf{k},\mathbf{b}} + \zeta^2_{\mathbf{k},-\mathbf{b}} +$$
$$\zeta_{\mathbf{k},\mathbf{b}}\,\zeta_{\mathbf{k},-\mathbf{b}}\,(e^{i\,\mathbf{k}\,.\,\mathbf{r}} + e^{-i\,\mathbf{k}\,.\,\mathbf{r}})] \tag{21}$$

and, after performing the calculations, we find

$$\mathscr{H}_{12} = 2\,[\,3\,/\,(4\,m\,\lambda) - G^2\,/\,(4\,\pi\,\sqrt{2\,\pi\,\lambda}\,)\,] + (\mathbf{b}\cdot\mathbf{b})$$
$$[\,m + G^2\,/\,(6\,\pi\sqrt{2\,\pi\,\lambda}\,)\,] - (G^2\,/\,4\,\pi)\,[\,1 + ½\,\mathbf{b}\cdot\mathbf{b} \tag{22}$$
$$-\,(\mu\,/\,2\,r)\,(\mathbf{b}\cdot\mathbf{r})^2 - (1/2\,r^2)\,(\mathbf{b}\cdot\mathbf{r})^2\,]\,(e^{-\mu r}\,/\,r)$$

We may notice, by comparison of this equation with eq. (15), that the first part of the previous expression is twice the nucleon energy. So, one can write for the interaction energy between the two nucleons

$$\mathscr{V}_{int} = -\,(G^2\,/\,4\,\pi)\;[\,1 + ½\,\mathbf{b}\cdot\mathbf{b} - (\mu\,/\,2\,r)\,(\mathbf{b}\cdot\mathbf{r})^2 - (1/2\,r^2)\,(\mathbf{b}\cdot\mathbf{r})^2\,]\,(e^{-\mu r}/\,r) \tag{23}$$

For systems described by the Schrödinger equation the Lagrangian is

$$\mathscr{L} = <\psi\,|\,i\,d/dt - H\,|\,\psi> \tag{24}$$

When we perform the calculation we obtain

$$<\psi\,|\,\partial/\partial\,\vec{\mathbf{r}}\,|\,\psi> = -\,i\,\mathbf{b}[\,m + G^2\,/\,(\,6\,\pi\sqrt{2\,\pi\,\lambda}\,)\,] \tag{25}$$

and, therefore, we can write

$$\mathscr{L} = \dot{\mathbf{r}}\cdot\mathbf{b}\,[\,m + G^2\,/\,(6\,\pi\sqrt{2\,\pi\,\lambda}\,)\,] - \mathscr{H}_{12} \tag{26}$$

where $\mathbf{r}$ represents the relative coordinate of the two particles and $\mathscr{H}_{1,2}$ is given by eq. (21).

From $\partial\mathscr{L}\,\partial\,/\,\vec{\mathbf{b}} = 0$ we conclude that

$$\dot{\mathbf{r}} = 2\,\mathbf{b} - (G^2\,/\,4\,\pi)\,[\,m + G^2\,/\,(6\,\pi\,\sqrt{2\,\pi\,\lambda}\,)\,]^{-1}\,[\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{r})\,(\mathbf{r}\,/\,r^2) - (\mathbf{b}\cdot\mathbf{r})\,(\mu\,\mathbf{r}\,/\,r)\,]\,(e^{-\mu r}\,/\,r) \tag{27}$$

Inspection of eq. (24) and (26) allows us to conclude that:

i) the relation between $\mathbf{r}$ and $\mathbf{b}$ is now given by eq. (27), so the relative velocity is equal to $2\,\mathbf{b}$ when $|\mathbf{r}| \to \infty$, as it should be;

ii) the momentum canonically conjugate to the relative distance between the two nucleons, $\mathbf{r}$, is

$$\mathbf{p} = [\, m + G^2 / (6\,\pi \sqrt{2\,\pi\,\lambda}\,) \,]\, \mathbf{b}$$

in agreement with the expression of the reduced mass for the relative motion of two identical interacting particles, which is half of the mass $\mathcal{M}$ given by eq. (17).

4 — CONCLUSIONS

The study of interacting nucleon systems by means of the exchange of scalar mesons was performed considering systems with one and two nucleons.

In the first case we obtain an expression for the energy of the system when the parametrization introduced is clearly defined.

The expression obtained for systems with two nucleons reduces to the Yukawa interaction for zero relative velocity and introduces new factors which depend on the relative velocity of the nucleons in the other cases.

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