THE ASYMPTOTIC APPROXIMATION IN HEAVY ION TRANSFER REACTIONS (*)

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ABSTRACT — Finite range effects in heavy ion transfer reactions are discussed using a previously developed formalism where the bound state wave functions are represented by the asymptotic spherical Hankel functions. Using the asymptotic approximation and a local recoil momentum a DWBA computer code was developed wich has the simplicity of a zero-range program and allows the inclusion of finite range and recoil effects. Good agreement is obtained with results from full finite range calculations for one nucleon transfer reactions. The effects of the Coulomb terms in the interaction are discussed.

1 — INTRODUCTION

Finite range and recoil effects are known to be generally very important in transfer reactions induced by heavy-ions. Various approximate methods [1]-[11] have been proposed for treating these effects in the distorted wave Born approximation (DWBA). By using them we obtain a more detailed understanding of the reaction mechanism, particularly of those aspects which are specifically related with the finite range of the interaction. Furthermore they are useful since full finite range calculations are time consuming, specially when it is necessary to include multi-step contributions to the reaction cross section.

Here we report on an extension of the Buttle and Goldfard approximation [1] in which the recoil effects associated with the

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finite mass of the transferred particle can be easily included. From another point of view our model for the transition amplitude is a generalization of the zero range approximation to states with orbital angular momentum different from zero. The reaction form factor, which may include the effects of recoil, is calculated using analytical methods and the DWBA calculation retains the simplicity of a zero-range code.

Results of calculations using this model for various reactions, including cases where the contribution from unnatural parity processes is important, are compared with full finite range calculations. The significance and reliability of the model is discussed. In the present calculation and also in those of a preceeding short paper [12] the recoil momentum is obtained using a local momentum approximation.

With the same model and approximations that are described here in detail we have performed calculations for the cross section and analysing powers of the ^{58}Ni(^{7}Li, ^{6}Li)^{59}Ni reaction at an incident energy of 20.3 MeV. In this work, published elsewhere [13], [14], it was shown that the agreement between the DWBA predictions and the observed Q-value dependence of the vector analysing power was greatly improved when the recoil momentum is generated using a semi-classical model proposed by Brink [15], [16].

2 — DWBA FORMALISM

The DWBA transition amplitude for the transfer reaction A(a,b)B where a = b + x and x is the transferred cluster is given by

$$T_{ba} = \int d\mathbf{r}_{xA}\; d\mathbf{r}_{bx}\; \chi_b^{(-)*}(\mathbf{r}_b) < Bb \,|\, V \,|\, Aa > \chi_a^{(+)}(\mathbf{r}_a), \qquad (1)$$

where the coordinates $\mathbf{r}_{xA}$, $\mathbf{r}_{bx}$, $\mathbf{r}_a$ and $\mathbf{r}_b$ are illustrated in Fig. 1 and $\chi_a^{(+)}$ and $\chi_b^{(-)}$ are elastic scattering wave functions in the entrance and exit channels. Using a fractional parentage expansion of the internal states of a and B into states of x and performing the integration over the internal coordinates of b,

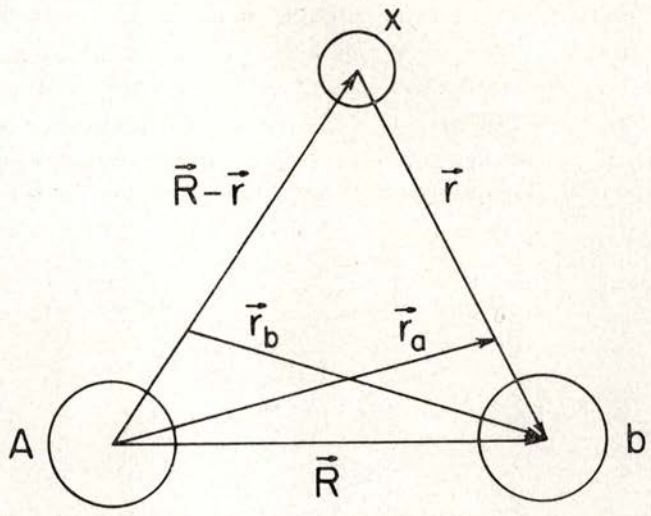

Fig. 1 — The coordinate vectors for a general transfer reaction.

A and x the transition matrix can be written, using essentially the notation of ref [10], as

$$< J_B M_B \; s_b \sigma_b | V | J_A M_A \; s_a \sigma_a > =$$

$$\sum_{L j_1 j_2} (J_A M_A \; j_2 \xi_2 | J_B M_B) \; (L\Lambda \; j_1 \xi_1 | j_2 \xi_2)$$

$$\cdot \; (-1)^{s_b - \sigma_b} (s_a \sigma_a \, s_b - \sigma_b | j_1 \xi_1) \hat{L} \sum_{l_1 l_2} A^{l_1 l_2}_{L j_1 j_2} \beta^{l_1 l_2 L\Lambda}_{j_1 j_2} , \qquad (2)$$

where $\hat{L} = (2L + 1)^{1/2}$,

$$A^{l_1 l_2}_{L j_1 j_2} = \Theta_{l_1 j_1} \; \Theta_{l_2 j_2} \; \hat{s}_a \hat{L} (-1)^{j_1 + s_x} \begin{Bmatrix} L & j_1 & j_2 \\ s_x & l_2 & l_1 \end{Bmatrix} , \qquad (3)$$

and the reduced amplitude β is given by

$$\beta^{l_1 l_2 L\Lambda}_{j_1 j_2} = \frac{i^{-L}}{\hat{L}} \int d\mathbf{r}_{xA} \; d\mathbf{r}_{bx} \; \chi^{(-)*}_b (\mathbf{r}_b) \; f^{l_1 l_2}_{L j_1 j_2 \Lambda} (\mathbf{r}_{xA}, \mathbf{r}_{bx}) \; \chi^{(+)}_a (\mathbf{r}_a) . \qquad (4)$$

Here L is the total orbital angular momentum transfer in the reaction and the spins of A, a, B, b and x are denoted by J_A, s_a, J_B, s_b and s_x. The quantities $\Theta_{l_1 j_1}$ and $\Theta_{l_2 j_2}$ are fractional parentage coefficients corresponding to a bound x with angular momentum quantum numbers $l_1 j_1$ and $l_2 j_2$. The form factor of the reduced amplitude is given by

$$f^{l_1 l_2}_{L j_1 j_2 \Lambda}(\mathbf{r}_{xA}, \mathbf{r}_{bx}) = i^{L+l_1-l_2} \sum_{\lambda_1 \lambda_2} (l_2 \lambda_2 l_1 \lambda_1 | L\Lambda)$$

$$\cdot R_{l_2 j_2}(r_{xA})\, Y^{\lambda_2 *}_{l_2}(\hat{\mathbf{r}}_{xA})\; V\, R_{l_1 j_1}(r_{bx})\, Y^{\lambda_1 *}_{l_1}(\hat{\mathbf{r}}_{bx}), \qquad (5)$$

where $R_{l_1 j_1}$ and $R_{l_2 j_2}$ are the normalized radial wave functions of x in the projectile and residual nucleus. In the conventional form of the DWBA the interaction V responsible for the transfer is assumed to be V_{bx} in the post representation and V_{Ax} in the prior representation.

It is well known that there is a strong localization of the integrand in eq. (4), as a function of the separation **R** between the heavy ion cores, due to combined action of the Coulomb barrier and of large absorption for small R, with the lack of overlap between the bound state wave functions at large R. It is therefore appropriate to treat the deviations from **R** in the arguments of the distorted waves in an approximate way. To discuss such approximations it is useful to perform Taylor expansions of $\chi^{(+)}_a$ and $\chi^{(-)}_b$. Representing $\mathbf{r}_{bx}$ by **r** we can write

$$\beta^{l_1 l_2 L\Lambda}_{j_1 j_2} = \frac{i^{-L}}{\hat{L}} \int d\mathbf{R}\, d\mathbf{r}\, e^{\mathbf{r}\cdot\mathbf{0}} f^{l_1 l_2}_{L j_1 j_2 \Lambda}(\mathbf{R}-\mathbf{r}, \mathbf{r})\, \chi^{(-)*}_b(\gamma\mathbf{R})\, \chi^{(+)}_a(\mathbf{R}), \qquad (6)$$

where $\gamma = M_A/M_B$,

$$\mathbf{0} = \frac{M_x}{M_A}\nabla_b - \frac{M_x}{M_a}\nabla_a , \qquad (7)$$

and ∇_a (∇_b) is the gradient with respect to $\mathbf{R}$ and acts only on the function $\chi_a^{(+)}$ ($\chi_b^{(-)*}$). Notice that we could have used $\mathbf{r}_{xA}$ instead of $\mathbf{r}_{bx}$ as integration variable. A similar derivation gives

$$\beta_{j_1 j_2}^{l_1 l_2 L\Lambda} = \frac{i^{-L}}{\hat{L}} \int d\mathbf{R}\, d\mathbf{r}\, e^{\mathbf{r}\cdot\mathbf{Q}} f_{Lj_1 j_2\Lambda}^{l_1 l_2}(\mathbf{r}, \mathbf{R}-\mathbf{r})\, \chi_b^{(-)*}(\mathbf{R})\, \chi_a^{(+)}(\mu\mathbf{R}), \tag{8}$$

where $\mathbf{r}_{xA}$ was also represented by $\mathbf{r}$, $\mu = M_b/M_a$ and

$$\mathbf{Q} = \frac{M_x}{M_b}\nabla_a - \frac{M_x}{M_B}\nabla_b . \tag{9}$$

Both eqs. (6) and (8) are exact and will be used as the starting point for our analysis.

When comparing eqs. (6) and (8) we notice that the vector $\mathbf{r}$ in the translation operator is the argument of the projectile and residual nucleus bound state wave functions, respectively. In relation with our model this makes the expression (6) more convenient to use in the post form of the DWBA transition amplitude while the expression (8) is more convenient to use in the prior form of the DWBA transition amplitude.

In eqs. (6) and (8) the effects due to recoil are described by the operators exp ($\mathbf{r}.\mathbf{0}$) and exp ($\mathbf{r}.\mathbf{Q}$). These operators become equal to the identity operator if we assume that the transferred particle has no mass. This approximation, known as the no-recoil approximation, is described in ref. [1]. We can take into account the recoil effects to all orders in our expansion of the translation operator by assuming a local momentum approximation when operating with exp ($\mathbf{r}.\mathbf{0}$) or exp ($\mathbf{r}.\mathbf{Q}$) on the distorted waves. In this approximation the operator $\mathbf{0}$ in eq. (7) is replaced by the recoil momentum,

$$\mathbf{p} = -\left(\frac{M_x}{M_B}\mathbf{k}_b + \frac{M_x}{M_a}\mathbf{k}_a\right), \tag{10}$$

where $\mathbf{k}_a$ and $\mathbf{k}_b$ represent local momenta in the entrance and exit channels, respectively. The usual procedure is to choose the direction of $\mathbf{p}$ along the bisector of the scattering angle corresponding to the distance of closest approach for a Coulomb orbit

in the exit channel. It has been shown by Braun-Munzinger et al. [17], that the reduced transition amplitude depends weakly on that direction.

We can follow a different approach in which the recoil momentum **p** is generated using a semiclassical model [15], [16]. This has been described in references [13], [14]. We obtain

$$\mathbf{p} = -\mathbf{n}\left(\frac{Q}{v} + \frac{1}{2} M_x v\right) \tag{11}$$

where **n** is a unit vector in the reaction plane tangent to the projectile trajectory at the point where the transfer is more probable, v is the relative velocity between the heavy ions at that point and Q is the reaction Q-value.

3 — USE OF SPHERICAL HANKEL FUNCTIONS

The most simple way to represent the bound states in a heavy ion transfer reaction is to use the Hankel function which describes the asymptotic behaviour for large r. In this approximation, which we shall call the asymptotic approximation, the form factor of the reduced amplitude in the post representation is

$$f^{l_1 l_2}_{Lj_1 j_2 \Lambda}(\mathbf{R}\text{-}\mathbf{r}, \mathbf{r}) = i^{L+l_1-l_2} \sum_{\lambda_1 \lambda_2} (l_2 \lambda_2 l_1 \lambda_1 | L\Lambda)\, N'_0\, i^{l_2} h^{(1)}_{l_2}(i\beta\,|\,\mathbf{R}\text{-}\mathbf{r}|)$$

$$Y^{\lambda_2 *}_{l_2}(\hat{\mathbf{R}\text{-}\mathbf{r}})\, \frac{\hbar^2}{2M_x\mu}\, (\nabla^2 - \alpha^2)\, N_0\, i^{l_1} h^{(1)}_{l_1}(i\alpha r)\, Y^{\lambda_1}_{l_1}(\hat{\mathbf{r}}), \tag{12}$$

where we have used the Schrödinger equation of nucleus a to replace the V_{bx} interaction and assumed $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Using a recoil momentum **p** the **r** integration in eq. (6) is given by

$$\int d\mathbf{r}\; e^{i\mathbf{p}\cdot\mathbf{r}}\, f^{l_1 l_2}_{Lj_1 j_2 \Lambda}(\mathbf{R}\text{-}\mathbf{r}, \mathbf{r}) = \frac{\hbar^2}{2M_x\mu}\, N'_0 N_0\, i^L\, (-1)^{l_1+l_2}$$

$$\cdot \sum_{\lambda_1 \lambda_2} (l_2 \lambda_2 l_1 \text{-}\lambda_1 | L\Lambda)\, (-1)^{\lambda_1} H_{l_1 \lambda_1 l_2 \lambda_2}(\mathbf{R}, \mathbf{p}, \alpha, \beta), \tag{13}$$

where

$$H_{l_1 \lambda_1 l_2 \lambda_2}(\mathbf{R}, \mathbf{p}, \alpha, \beta) = \int d\mathbf{r}\, [h^{(1)}_{l_2}(i\beta|\mathbf{R}-\mathbf{r}|)\, Y^{\lambda_2}_{l_2}(\hat{\mathbf{R}-\mathbf{r}})]^* \cdot (\nabla^2 - \alpha^2)\, h^{(1)}_{l_1}(i\alpha r)\, Y^{\lambda_1}_{l_1}(\hat{\mathbf{r}})\, e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{14}$$

The procedure which is usually followed to calculate this type of integral is to separate variables in $h^{(1)}_{l_2}(i\beta|\mathbf{R}-\mathbf{r}|)\, Y^{\lambda_2}_{l_2}(\hat{\mathbf{R}-\mathbf{r}})$ using the well known addition theorem for Hankel functions [1], [18]. This, however, involves an expansion that in general does not converge uniformly and requires further approximations in the calculation of the folding integral. As shown in the Appendix 3 of ref [10] the intergral (14) can be calculated analitically without the above shortcomings. Using this result and eqs. (6), (13) and (14) we finally obtain for the reduced amplitude in the asymptotic approximation

$$\beta^{l_1 l_2 L\Lambda}_{j_1 j_2} = \frac{4\pi}{\hat{L}} \sum_{\substack{l_r \lambda_r \\ L_c \Lambda_c}} (l_r \lambda_r L_c \Lambda_c | L\Lambda) \int d\mathbf{R}\, Y^{\Lambda_c *}_{L_c}(\mathbf{R})\, y^{\lambda_r *}_{l_r}(i\mathbf{p}) \cdot F_{l_1 l_2 L L_c l_r}(R, p^2)\, \chi^{(-)*}_b(\gamma\mathbf{R})\, \chi^{(+)}_a(\mathbf{R}), \tag{15a}$$

where

$$F_{l_1 l_2 L L_c l_r}(R, p^2) = \Gamma_{l_1 l_2 L_r L L_c l_r} X_{l_1 l_2 L_r L_c l_r}(R) \tag{15b}$$

$$\Gamma_{l_1 l_2 L_r L L_c l_r} = (-1)^{L_r} \hat{l}_1 \hat{L}_r \hat{L}_c \hat{l}_r \begin{Bmatrix} l_1 & l_2 & L \\ L_c & l_r & L_r \end{Bmatrix} (l_1 0\, l_r 0 | L_r 0)\, (L_r 0\, L_c 0 | l_2 0), \tag{15c}$$

$$X_{l_1 l_2 L_r L_c l_r} = \frac{\hbar^2}{2M_x \mu} \frac{(2l_1+1)!!}{(2l_r+1)!!(2L_r+1)!!} N_0 N_0' \frac{\beta^{L_r}}{\alpha^{l_1+1}}\, i^{L_c}\, h^{(1)}_{L_c}(i\beta R), \tag{15d}$$

$(2l+1)!! = (2l+1)\ (2l-1) \ldots$ and

$$L_r = l_1 - l_r \ . \tag{16}$$

The orbital angular momenta l_r and L_c have a very simple physical interpretation [11]: l_r is the part of the total balance of orbital angular momentum in the transfer process which is due to the

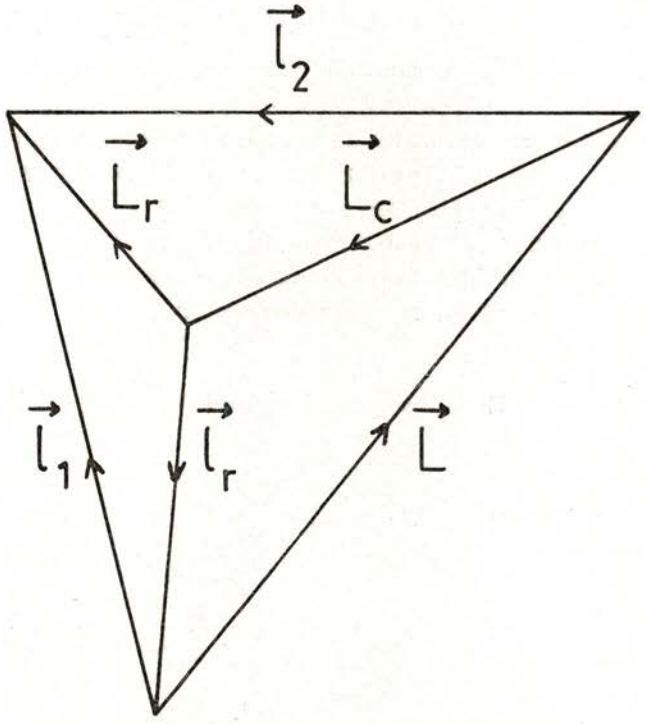

Fig. 2 — Vector coupling of the orbital angular momentum operators in a transfer reaction. The vectors are defined in the text.

finite mass of x and is usually called the recoil angular momentum. From the angular momentum coupling in eqs (15) we conclude that L_c is the orbital angular momentum transferred from the relative motion of the heavy ion cores A and b in the entrance channel to the relative motion of A and b in the exit channel. The coupling of orbital angular momentum in a transfer reaction is illustrated in Fig. 2, where the sum $l_r + l_1$ (equal to

$l_2 - L_c$) is represented by L_r. The no-recoil approximation is obtained with $l_r = 0$, which implies $L_c = L$ and $l_1 = L_r$. With this approximation the eqs. (15) reproduce the radial form factor of ref. [1],

$$F_{l_1 l_2 L L_c l_r}(R, p^2) = (-1)^{l_1} \hat{l}_1 \, (l_1 0 L 0 | l_2 0) \, \frac{\hbar^2}{2M_x \mu} \cdot N_0 N_0' \, \frac{\beta^{l_1}}{a^{l_1+1}} \, i^L \, h_L^{(1)}(i\beta R) . \tag{17}$$

This result shows that the present model can be considered as a natural extension of the Buttle and Goldfard approximation [1].

The most general selection rule due to parity conservation in the reaction involves the quantum numbers L_c and l_r. The particular form of angular momentum coupling in eq. (15c) implies that [11]

$$l_r + L_c + l_1 + l_2 = \text{even}. \tag{18}$$

In the exact DWBA matrix element there is no restriction on the values of l_r. However due to the small mass of the transferred cluster relative to the heavy ion cores only the first few values of l_r are likely to give important contributions to the transition amplitude. This has been shown, for instance, in the work of Braun-Munzinger et al. [17]. In the asymptotic approximation there is an upper limit for the allowed values of l_r, since according with eq. (16) $l_r \leq l_1$. Eq. (16) means that l_r is antiparallel to l_1. This selection rule is a consequence of our description of the bound state wave function of x in the projectile as one Hankel function. With this approximation we assume that the transfer process is insensitive to the finite range of the binding potential of x in a. In fact for one Hankel function the product $V_{bx} \, R_{l_1 j_1} \, Y_{l_1}^{\lambda_1}$, present in the form factor, has zero range (see eq. (A.23) of ref. [10]) and therefore it cannot generate an orbital angular momentum larger than l_1. For a finite range interaction we can have $l_r > l_1$. Notice that the asymptotic approxi-

mation allows for the inclusion of recoil effects and at the same time maintains the radial form factor X, for each l_r, proportional to $N_0 N_0'$. It is therefore particularly suited to study the dependence of the cross section on $N_c N_0'$ which is a well defined quantity in heavy ion transfer reactions, particularly at sub-Coulomb energies [19], [20].

Following essentially the same type of derivation we can readily obtain from eq. (8) the reduced transition amplitude for the asymptotic approximation in the prior form

$$\beta^{l_1 l_2 L \Lambda\,(\text{prior})}_{j_1 j_2} = (-1)^{l_1+l_2+L} \frac{4\pi}{\hat{L}} \sum_{\substack{l_r \lambda_r \\ L_c \Lambda_c}} (l_r \lambda_r L_c \Lambda_c | L\Lambda) \int d\mathbf{R}\; Y^{\Lambda_c *}_{L_c}(\hat{\mathbf{R}})$$

$$\cdot\; y^{\lambda_r *}_{l_r}(i\mathbf{p})\; F^{(\text{prior})}_{l_1 l_2 L L_c l_r}(R, p^2)\; \chi^{(-)*}_b(\mathbf{R})\; \chi^{(+)}_a(\mu\mathbf{R}), \quad (19a)$$

where

$$F^{(\text{prior})}_{l_1 l_2 L L_c l_r}(R, p^2) = \Gamma_{l_1 l_2 L_r L L_c l_r} X^{(\text{prior})}_{l_1 l_2 L'_r L_c l_r}(R), \quad (19b)$$

$$X^{(\text{prior})}_{l_1 l_2 L_r L_c l_r} = (-1)^{l_r} \frac{\hbar^2}{2M_{x\gamma}} \frac{(2l_2+1)!!}{(2l_r+1)!!\,(2L'_r+1)!!}$$

$$N_0 N_0' \frac{\alpha^{L'_r}}{\beta^{l_2+1}} i^{L_c} h^{(1)}_{L_c}(i\alpha R), \quad (19c)$$

and

$$L'_r = l_2 - l_r. \quad (19d)$$

This selection rule, involving the angular momentum $\mathbf{L}'_r = \mathbf{l}_2 - \mathbf{l}_r$ (equal to $\mathbf{l}_1 + \mathbf{L}_c$), implies that in the prior representation $l_r \leq l_2$.

We emphasize that as regards the bound states the asymptotic approximation is equivalent to a generalization to states with $l_1 > 0$ of what is usually called a zero range approximation in a transfer from an s-state. In fact notice that for $l_1 = 0$ the expression (12) implies that V_{bx} is proportional to a δ-function.

4 — COULOMB EFFECTS

In the preceeding analysis we have assumed that the interaction V in the DWBA matrix element is purely nuclear. Here we consider the effects of the Coulomb terms of this interaction. In the post representation these Coulomb terms are of the form [21]

$$\Delta V_c = \frac{Z_b Z_A e^2}{R} - \frac{Z_b Z_B e^2}{r_b}, \tag{20}$$

where Z_i is the charge of nucleus i and $\mathbf{r}_b = \gamma\mathbf{R} + (1-\gamma)\mathbf{r}$. Since ΔV_c has a weak dependence on $\mathbf{r}$ ($\gamma \approx 1$) we can write

$$\Delta V_c \approx \frac{Z_b Z_A e^2}{R}(1 - 1/\gamma) - \frac{Z_x Z_B e^2}{\gamma R}. \tag{21}$$

In the prior representation the analogous approximation yields

$$\Delta V_c^{(\text{prior})} \approx \frac{Z_b Z_A e^2}{R}(1 - 1/\mu) - \frac{Z_x Z_A e^2}{\mu R}. \tag{22}$$

The calculation of the effects of ΔV_c on the transition matrix is straightforward using eqs. (21) and (22) although it involves integrals of the type

$$J_{l_1 \lambda_1 l_2 \lambda_2}(\mathbf{R}, \mathbf{p}, \alpha, \beta) = \int d\mathbf{r}\, [h^{(1)}_{l_2}(i\beta|\mathbf{R}-\mathbf{r}|)\, Y^{\lambda_2}_{l_2}(\hat{\mathbf{R}-\mathbf{r}})]^* \cdot h^{(1)}_{l_1}(i\alpha r)\, Y^{\lambda_1}_{l_1}(\hat{\mathbf{r}})\, e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{23}$$

These integrals can be easily calculated for $\mathbf{p} = 0$ and were discussed in ref. [10]. For $\mathbf{p} \neq 0$ the analytic calculation of $J_{l_1 \lambda_1 l_2 \lambda_2}$ is considerably more difficult as it is shown in the Appendix, where an approximate expression is also derived.

5 — NUMERICAL RESULTS

A DWBA computer program called HYDRA [22] has been coded in FORTRAN which generates the distorted waves and calculates the reduced transition amplitudes using eqs. (15) and (19) in the post and prior representations. To illustrate the validity of the asymptotic approximation we performed calculations for angular distributions which were previously analyzed with other DWBA codes in particular with full finite range programs. Fig. 3 shows results for the one neutron transfer reaction $^{12}C(^{14}N, ^{13}N)^{13}C_{g.s}$ at $E_{Lab} = 78$ MeV obtained using the same optical model parameters as in ref. [24]. Since this is a transfer between two $p\,1/2$ states the recoil effects can be quite large. The calculations of Fig. 3 compare the usual no-recoil Buttle and Goldfard approximation with the asymptotic approximation described in this work and given by eqs. (15) and (19). We emphasize that in both cases the bound state wave functions are described by one Hankel function and depend exclusively on the parameters N_0, α_0 and N_0', β_0. The constants N_0 and N_0' were extracted from bound state wave function generated in a Woods-Saxon well with geometry parameters $r_0 = 1.25$ fm and $a = 0.65$ fm.

We find that in this reaction the asymptotic approximation provides an accurate description of the recoil effects. This can be seen in Fig. 4 where it is compared with full finite range results obtained with the code LOLA for the same optical model and bound state parameters and for incident energies of 78 and 100 MeV. Both curves agree well in shape and in absolute value. The product of spectroscopic factors for the initial and final states $S_1 S_2$ differs by less than 18 % at both energies. In fact we obtain in the post representation $S_1 S_2 = 0.50$ and 0.60 at 78 and 100 MeV, while the full finite range calculations [23], [26] give $S_1 S_2 = 0.53$ and 0.51, respectively. The result of DWBA calculations using the code BRUNHILD [17] are also shown in Fig. 4 and give $S_1 S_2 = 0.47$ and 0.44 at 48 and 100 MeV respectively.

Calculations using the asymptotic approximation for the proton transfer reaction $^{12}C(^{14}N, ^{13}C)^{13}N$ at 78 MeV are shown in Fig. 5. Good agreement is obtained in shape and magnitude

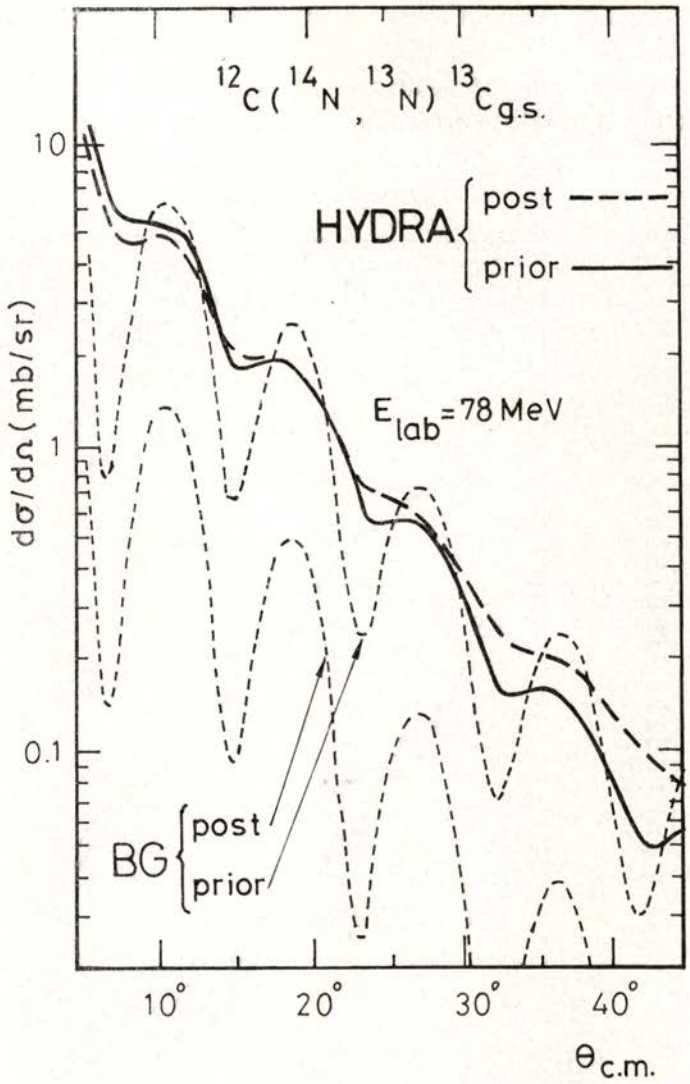

Fig. 3 — Comparison of the DWBA curves for the ^{12}C (^{14}N, ^{13}N) $^{13}C_{g.s.}$ reaction at $E_{Lab} = 78$ MeV obtained with HYDRA using the asymptotic approximation and with the no-recoil Buttle and Goldfard (BG) approximation in the post and prior representations. The optical potentials correspond to the set 3 of ref. [24].

with full finite range calculations [23], [25] performed with the same optical model and bound state parameters. These results indicate that the degree of convergence of the sum over the

recoil angular momentum l_r is well described by the selection rule $l_r \leq l_1$.

A test of the dependence of our results on the recoil momentum **p** was performed for the same reaction as in Fig. 4.

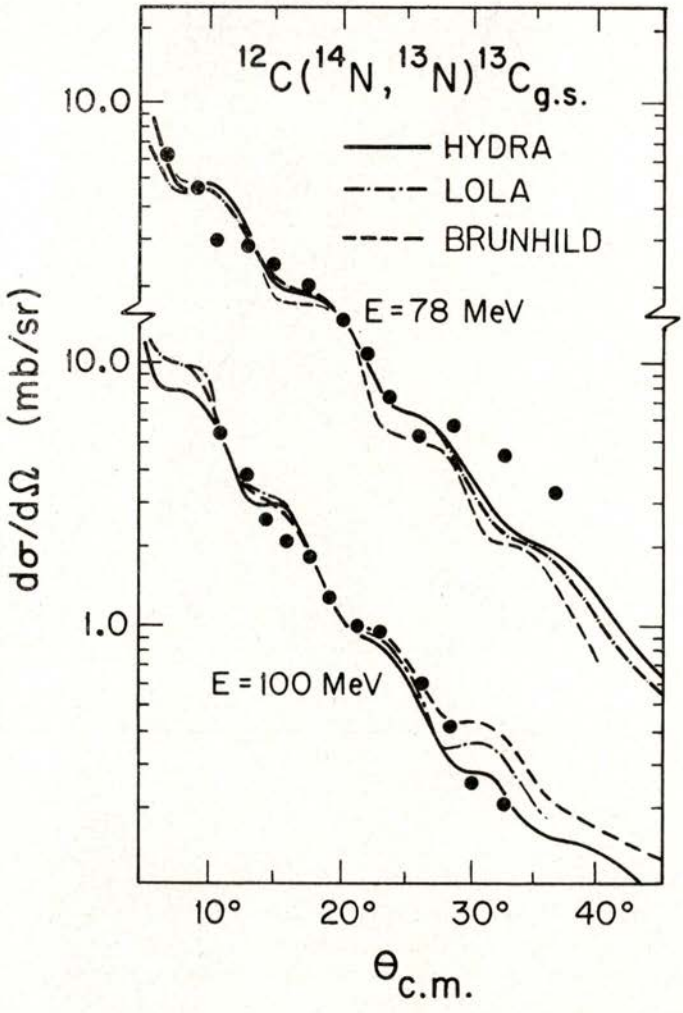

Fig. 4 — Comparison of the DWBA curves for the ^{12}C (^{14}N, ^{13}N) $^{13}C_{g.s.}$ reaction at incident energies of 78 and 100 MeV obtained, with the code HYDRA (full curve) using the asymptotic approximation, with the code LOLA [23], [26] (point-dash curve) and with the code BRUNHILD [17] (dash curve).

Using the local momentum approximation we find that the differential cross section is almost insensitive the modulus of **p**. The dependence on the direction of **p** is small and weaker in the post representation than in the prior representation. This is probably due to the presence of the $(-1)^{l_r}$ factor in eq. (19c), which for the dominant recoil term in this reaction is -1.

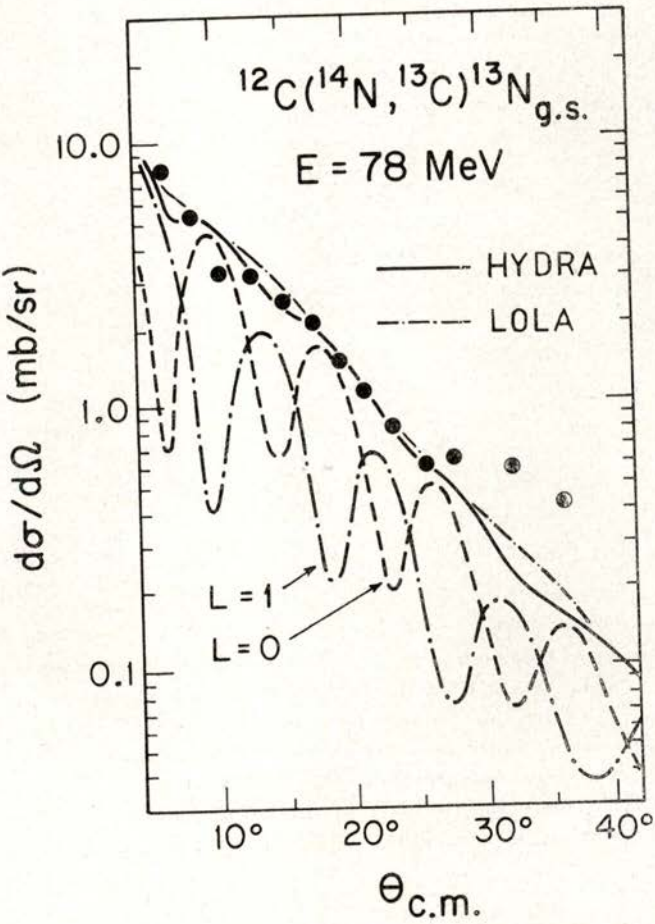

Fig. 5 — Same as in Fig. 5 for the ^{12}C (^{14}N, ^{13}C) $^{13}N_{g.s.}$ reaction at 78 MeV. The dashed curves show the angular distributions of the L = 0 and L = 1 transitions separately calculated with HYDRA. The sum is given by the full curve. The experimental values are from ref. [24].

In order to study the relative importance of recoil effects in the contributions from different values of the total orbital angular momentum L we have chosen the $^{26}Mg\,(^{16}O, ^{15}N)\,^{27}Al_{g.s.}$ reaction at $E_{Lab} = 45$ MeV since l_1 is different from l_2 and also because our results can be compared with previous full finite range DWBA calculations [27]. This is a transfer from a $p_{1/2}$ state to a $d_{5/2}$ state and therefore L = 2,3. The DWBA curves of Fig. 6

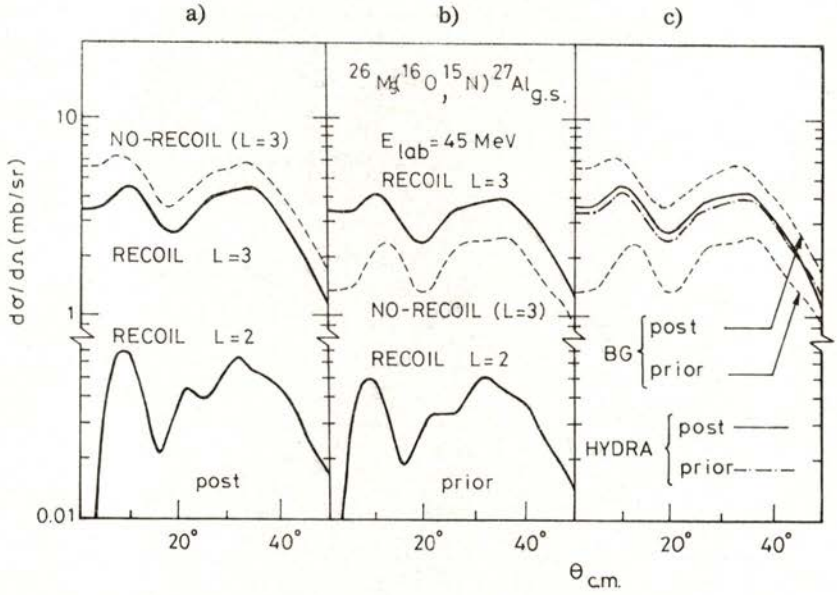

Fig. 6 — DWBA calculations for the $^{26}Mg\,(^{16}O, ^{15}N)\,^{27}Al_{g.s.}$ reaction at 45 MeV. Part a) shows a comparison in the post representation between the contribution from L = 2 and L = 3 calculated with HYDRA using the asymptotic approximation and the cross section calculated in the no-recoil Buttle and Goldfard (BG) approximation which only allows L = 3. Part b) shows the same comparison in the prior representation. Part c) shows the total cross section predicted by HYDRA and by the no-recoil BG approximation in both representations. The optical potentials are from ref. [27].

were obtained using the asymptotic approximation and the same optical model and bound state parameters as in ref. [27]. We find that the effects of recoil is either to increase or to decrease the cross section whether it is calculated in the prior or post

representations, respectively. As a result the discrepancy between the post and prior representations in the no-recoil approximation is strongly reduced. Fig. 7 shows that there is good agreement between the asymptotic and the full finite range calculations of Buttle [27] using the code DAISY for both $L = 2$ and $L = 3$

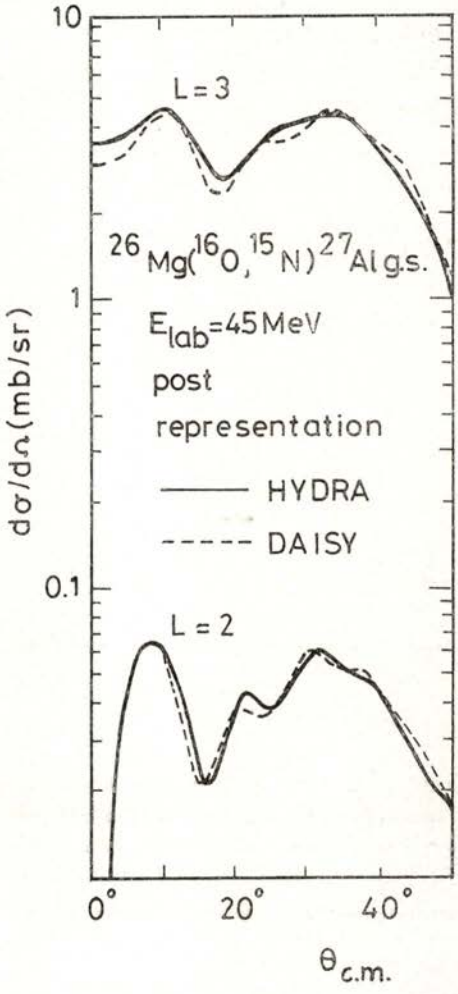

Fig. 7 — DWBA calculations for the ^{26}Mg (^{16}O, ^{15}N) $^{27}Al_{g.s.}$ reaction at 45 MeV. Comparison between calculations obtained with the code HYDRA (full curve) and with the full finite range code DAISY [27] (broken curve), for the $L = 2$ and $L = 3$ in the post representation.

although there is a factor of about 2 orders of magnitude between these two contributions to the cross section. The difference in the absolute magnitude of the differential cross section at the main peak is less than 10 % in the two calculations.

In the $^{26}Mg\,(^{16}O, ^{15}N)\,^{27}Al$ reaction the selection on the recoil angular momentum is $l_r \leq 1$ and $l_r \leq 2$ in the post and prior representations, respectively. This difference however does not have a marked effect on the cross section, because the contribution from $l_r = 2$ is very small, as shown in Fig. 8. Thus

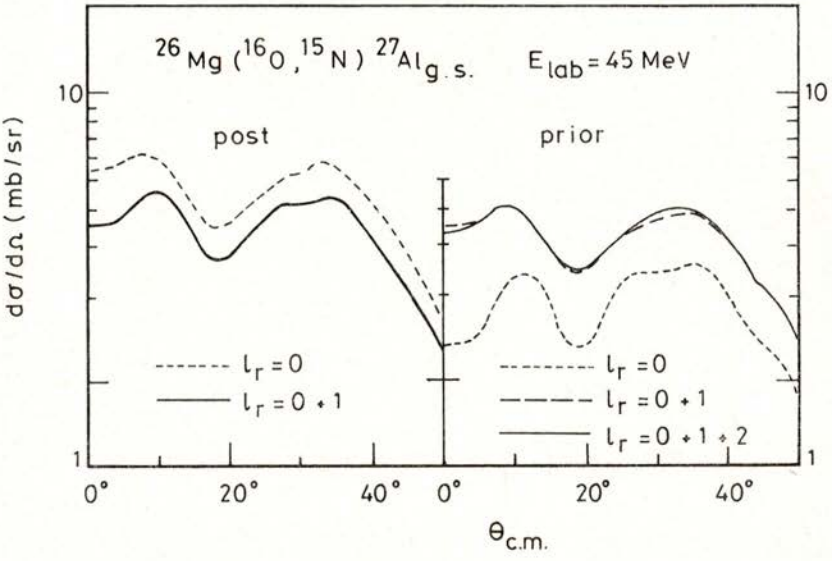

Fig. 8 — DWBA calculations for the same reaction as in Fig. 6 and 7 with HYDRA using the asymptotic approximation. The left hand side shows the contributions to the cross section in the post representation from the allowed values of $l_r = 0$ and 1. The right hand side shows the analogous contributions in the prior representation where l_r varies from 0 to 2.

although l_1 is different from l_2 the degree of convergence of the sum over l_r is well described by the asymptotic selection rule in both the prior and post representations.

5 — DISCUSSION

Using the formalism presented in ref. [10] for the approximate treatment of finite range and recoil effects a DWBA program was developed and applied to the analysis of quasi-elastic heavy ion transfer reactions.

In our approach the DWBA calculation in reduced to a zero-range type calculation since the form factor, wich may include the effects of recoil, is calculated analitically. The simplification is obtained through the representation of the bound state wave functions by the asymptotic spherical Hankel functions. This representation provides an accurate description of the reaction due to the strong localization of the transfer process outside the nuclear surfaces. In this asymptotic approximation the cross section is proportional to the product of the squares of the normalization of the tail of the bound state wave functions wich is a well defined quantity in heavy ion transfer reactions [19], [20].

We obtain good agreement with the results of full finite range calculations [23], [25], [26], [27] in all one-nucleon transfer reactions investigated so far, using considerably smaller computing times. Also we find that the inclusion in the asymptotic approximation of the Coulomb terms, that are present in the interaction responsible for the transfer, improves considerably the agrement with the spectroscopic factors extracted from full finite range calculations.

Our approximation can be expected to break down in cases where a substantial part of the cross section comes from the nuclear interior and also in multinucleon transfer reactions where the local momentum approximation may be inadequate to account for the recoil effects. Within its region of applicability the present approach gives a reliable description of finite range and is particulary simple to use. It can be useful in coupled reaction channel calculations to include finite range and recoil effects with a very small increase in the computing time.

From another point of view we note that the integral on the left hand size of eq. (13), which gives the reaction form factor in the DWBA, is formally similar to the transfer amplitude in

the Brink semi-classical model of transfer reactions (compare for instance with eq. (6.16) of ref. [15]). This means that the asymptotic approximation can also be applied in the context of semi-classical models of transfer reactions. Finally we note that the development of approximate methods to deal with finite range and recoil effects is made particularly significant by recent results [28] where it is shown that the usual form of the DWBA is often unable to interpret the transfer reaction data.

We wish to thank Dr. D. M. Brink, Dr. P. J. A. Buttle and Dr. R. C. Johnson for many interesting and valuable discussions during the course of this work. The computer time provided at the IBM-360/44 of the Centro de Cálculo das Universidades de Lisboa, where the calculations were done, is gratefully acknowledged.

APPENDIX

FOLDING INTEGRAL WITH TWO HANKEL FUNCTIONS

The integral

$$J_{l_1 \lambda_1 l_2 \lambda_2}(\mathbf{R}, \mathbf{p}, \alpha, \beta) = \int d\mathbf{r}\, [h_{l_1}^{(1)}(i\beta|\mathbf{R}-\mathbf{r}|)\, Y_{l_2}^{\lambda_2}(\hat{\mathbf{R}-\mathbf{r}})]^* \cdot h_{l_1}^{(1)}(i\alpha r)\, Y_{l_1}^{\lambda_2}(\hat{\mathbf{r}})\, e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{A.1}$$

can be written in momentum space as

$$J_{l_1 \lambda_1 l_2 \lambda_2}(\mathbf{R}, \mathbf{p}, \alpha, \beta) = \int d\mathbf{k}\, e^{i\mathbf{R}\cdot\mathbf{k}} F_{l_2 \lambda_2}(\mathbf{k}, \beta)^*\, F_{l_1 \lambda_1}(\mathbf{p}-\mathbf{k}, \alpha), \tag{A.2}$$

where $F_{l\lambda}(\mathbf{k}, \beta)$ is the Fourier transform of $h_l^{(1)}(i\beta r)\, Y_l^{\lambda}(\hat{\mathbf{r}})$. The calculation of this integral is considerably simplified if we neglect the dependence on the angle between $\mathbf{k}$ and $\mathbf{p}$ in the

denominator of the radial part of $F_{l_1 \lambda_1}$. Assuming **p** perpendicular to **k** we get

$$F_{l_1 \lambda_1}(\mathbf{p}\text{-}\mathbf{k}, \alpha) \cong -\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\alpha^{l_1+1}} \frac{1}{\alpha_p^2+k^2} Y_{l_1}^{\lambda_1}(\mathbf{p}\text{-}\mathbf{k}). \quad (A.3)$$

where $\alpha_p = (p^2 + \alpha^2)^{1/2}$. This approximation is reasonable as long as p is sufficiently smaller than α. Otherwise we can use the expansion

$$\frac{1}{(\mathbf{p}\text{-}\mathbf{k})^2 + \alpha^2} = \frac{4\pi}{2pk} \sum_{lm} Q_l(z)\, Y_l^{m\,*}(\hat{\mathbf{p}})\, Y_l^m(\hat{\mathbf{k}}). \quad (A.4)$$

where Q_l are Legendre functions of the second kind and

$$z = (p^2 + k^2 + \alpha^2)/2pk.$$

Using the well known addition theorem

$$Y_{l_1}^{\lambda_1}(\mathbf{p}\text{-}\mathbf{k}) = \sum_{l_r} a_{l_1 l_r}(l_r \lambda_r L_r \Lambda_r | l_1 \lambda_1)\, Y_{l_r}^{\lambda_r}(\mathbf{p})\, Y_{L_r}^{\Lambda_r}(-\mathbf{k}) \quad (A.5)$$

where $L_r = l_1 - l_r$,

$$a_{l_1 l_r} = (4\pi)^{1/2} (-1)^{l_r} \hat{l}_1 \frac{(2l_1+1)!!}{(2l_r+1)!!(2L_r+1)!!} (l_1 0\, l_r 0 | L_r 0) \quad (A.6)$$

and eqs. (A.2) and (A.3), we can write

$$J_{l_1 \lambda_1 l_2 \lambda_2}(\mathbf{R}, \mathbf{p}, \alpha, \beta) \simeq \sum_{l_r} a_{l_1 l_r}(l_r \lambda_r L_r \Lambda_r | l_1 \lambda_1) \frac{\alpha_p^{L_r+1}}{\alpha^{l_1+1}} Y_{l_r}^{\lambda_r}(\mathbf{p})$$

$$\cdot \int d\mathbf{k}\, e^{i\mathbf{R}\cdot\mathbf{k}}\, F^*_{l_2 \lambda_2}(\mathbf{k}, \beta)\, F_{L_r \Lambda_r}(\mathbf{k}, \alpha_p). \quad (A.7)$$

This integral is given in eq. (A.36) of Ref. [10]. With this result we obtain

$$J_{l_1 \lambda_1 l_2 \lambda_2}(\mathbf{R}, \mathbf{p}, \alpha, \beta) =$$

$$\sum_{l_r \lambda_r L_c \Lambda_c L} i^{l_2 - l_1 + L_c} (-1)^{\lambda_1} (l_2 \lambda_2 l_1 - \lambda_1 | L\Lambda)(l_r \lambda_r L_c \Lambda_c | L\Lambda)$$

$$\cdot Y_L^{\Lambda_c *}(\hat{\mathbf{R}})\, Y_{l_r}^{\lambda_r *}(i\mathbf{p})\, \Gamma_{l_1 l_2 L_r L L_c l_r} \frac{(2l_1+1)!!}{(2l_r+1)!!(2L_r+1)!!}$$

$$\cdot \frac{\alpha_p^{L_r+1}}{\alpha^{l_1+1}} f_{L_r L_c l_2}(R, \alpha_p, \beta). \qquad \text{(A.8)}$$

where $L_r = l_1 - l_r$, $\Gamma_{l_1 l_2 L_r L L_c l_r}$ has been defined in eq. (28c) and

$$f_{L_r L_c l_2}(R, \alpha_p, \beta) = \frac{1}{\alpha_p^2 - \beta^2}\left[\frac{\alpha_p^{l_2}}{\beta^{l_2+1}} h_{L_c}^{(1)}(i\alpha_p R) - \frac{\beta^{L_r}}{\alpha_p^{L_r+1}} h_{L_c}^{(1)}(i\beta R)\right]. \qquad \text{(A.9)}$$

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