INSTABILITY SATURATION BY NONLINEAR MODE COUPLING IN A PLASMA

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ABSTRACT — We study the instability saturation by the process of resonant three-wave coupling, comparing the results obtained in the fixed and random phase approximations. We show that the transition to chaos occurring in the fixed phase equations does not exist in the random phase equations. In quite general conditions the random phase leads to a well defined saturation level of the unstable mode.

1 — INTRODUCTION

It is now commonly accepted in Plasma Physics that the turbulent state arises from some instability which eventually saturates and breaks into a number of different modes. In this context, an important problem is the determination of the nonlinear state resulting from a linearly unstable wave.

One of the elementary processes leading to the instability saturation is the resonant three-wave coupling. The equations describing the interaction of three waves have been already studied, assuming that the two linearly damped modes into which the unstable mode decays are equal and further assuming that the unstable wave is strictly monochromatic [1], [2]. It has been shown numerically that in this case a strange atractor occurs and the wave amplitudes can behave chaotically. This fixed phase approximation is only valid when the frequency width of the unstable wave packet, Δ, is less than $1/\tau_c$, where τ_c is the characteristic time for the instability saturation.

In the present work we discuss the instability saturation from a more general point of view. In Section 2 we state the nonlinear equation for the wave field amplitude in its general form. In Section 3 we discuss the fixed phase approximation which can be used in order to solve the nonlinear equation. Section 4 is devoted to the analysis of the random phase approximation, which is valid when $\Delta >> 1/\tau_c$. Finally in Section 5 we state the conclusions, and compare the results obtained in the fixed and random phase approximations.

2 — NONLINEAR EQUATION

Let us assume the general situation of an infinite, homogeneous and nonlinear dielectric medium, which can be a plasma. Using Maxwell equations and making space and time Fourier transforms we can get the equation of propagation for the electric field $E(\mathbf{k})$ in the following form:

$$D(\mathbf{k}, \omega)\; E(\mathbf{k}) = -(i/\omega\,\varepsilon_0)\; \hat{\mathbf{a}}_k^* \cdot \mathbf{J}_{NL}(\mathbf{k}) \qquad (1)$$

where

$$D(\mathbf{k}, \omega) = \frac{c^2}{\omega^2}\,|\mathbf{k}\cdot\hat{\mathbf{a}}_k|^2 - \frac{k^2 c^2}{\omega^2} + \hat{\mathbf{a}}_k^* \cdot \overline{\overline{\varepsilon}}(\mathbf{k}, \omega)\cdot\hat{\mathbf{a}}_k \qquad (2)$$

In these equations $\overline{\overline{\varepsilon}}(\mathbf{k}, \omega)$ is the dielectric tensor describing the medium, ω and $\mathbf{k}$ are the frequency and wavevector associated to the field $E(\mathbf{k})$ and $\hat{\mathbf{a}}_k$ is the unit polarization vector. The nonlinear current appearing in equation (1) is given by:

$$\mathbf{J}_{NL}(\mathbf{k}) = \int \overline{\overline{\overline{\sigma}}}(\mathbf{k}, \mathbf{k}')\cdot\mathbf{E}(\mathbf{k}')\;\mathbf{E}(\mathbf{k}'')\; e^{i\Omega t}\,\frac{d\mathbf{k}'}{(2\pi)^3} \qquad (3)$$

where $\overline{\overline{\overline{\sigma}}}(\mathbf{k}, \mathbf{k}')$ is the 'second order conductivity' tensor, $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$ and Ω is the frequency mismatch

$$\Omega = \omega(\mathbf{k}) - \omega'(\mathbf{k}') - \omega''(\mathbf{k}'') \qquad (4)$$

We see from equation (1) that when the nonlinear current is neglected, $\mathbf{J}_{NL}(\mathbf{k}) = 0$, a nontrivial solution for the electric wave field $E(\mathbf{k})$ implies that:

$$D(\mathbf{k}, \omega) = 0 \tag{5}$$

This is the linear dispersion relation which specifies ω as a function of $\mathbf{k}$, $\omega = \omega(\mathbf{k})$. Equation (4) shows that, in the expression for the nonlinear current (3) we assume that each Fourier component of the electric field in the medium $E(\mathbf{k})$ obeys the linear dispersion equation (5). However, if the nonlinear current is taken into account, $\mathbf{J}_{NL}(\mathbf{k}) \neq 0$, equation (5) is no longer valid. Developping $D(\mathbf{k}, \omega)$ around the linear dispersion relation we get [3]:

$$D(\mathbf{k}, \omega) \simeq i\left(\frac{\partial D}{\partial \omega}\right)_k \left[\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \frac{\partial}{\partial \mathbf{r}} + \gamma_k\right] \tag{6}$$

where $\mathbf{v}_k = \partial \omega / \partial \mathbf{k}$ is the linear group velocity and γ_k is the linear damping coefficient for the wave field $E(\mathbf{k})$. Replacing (6) in equation (1) we get the nonlinear equation in its final form:

$$\begin{aligned}(2\pi)^3 \; (d/dt + \gamma_k) \; E(\mathbf{k}) \\ = - \int H(\mathbf{k}, \mathbf{k}') \; E(\mathbf{k}') \; E(\mathbf{k}-\mathbf{k}') \; e^{i\Omega t} \; d\mathbf{k}'\end{aligned} \tag{7}$$

where the total time derivative means:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_k \cdot \frac{\partial}{\partial \mathbf{r}} \tag{8}$$

and the nonlinear coupling coefficient $H(\mathbf{k}, \mathbf{k}')$ is given by:

$$H(\mathbf{k}, \mathbf{k}') = \frac{\hat{\mathbf{a}}_k^* \cdot \overline{\overline{\overline{\sigma}}}(\mathbf{k}, \mathbf{k}') \cdot \hat{\mathbf{a}}_{k'} \hat{\mathbf{a}}_{k''}}{\varepsilon_0 \, \omega \, (\partial D / \partial \omega)_k} \tag{9}$$

Equation (7) gives the rate of change of the Fourier component $E(\mathbf{k})$ of the electric field in the medium due to the nonlinear three-wave interaction of $E(\mathbf{k})$ with each of the pairs $E(\mathbf{k}')$ and $E(\mathbf{k}'')$ of the spectrum which satisfy the selection rule: $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$. The solution of this equation would give the turbulent spectrum in a plasma if the three-wave interaction was the only process of energy exchange between the Fourier components of the spectrum. Actually, in a turbulent plasma, not only the four-wave or the higher order wave interaction processes are present but also are the wave-particle processes which cannot be described in the frame of the dielectric theory used here.

3 — FIXED PHASE APPROXIMATION

Let us apply the general nonlinear equation (7) to the case of instability saturation. The physical picture is the following: A given region of the spectrum around $\mathbf{k} = \mathbf{k}_1$ becomes unstable, due to the linear properties of the plasma. In this case we have $\gamma_1 = \gamma_{k_1} < 0$. If the unstable wave field attains a significant level, the nonlinear effect described by the right hand side of equation (7) is able to transfer a significant amount of energy from the unstable region of the spectrum to the stable regions where dissipation occurs. Instability saturation can then be achieved.

In order to discuss this process in detail it is useful to assume that the unstable spectrum reduces to a single wave $\mathbf{k} = \mathbf{k}_1$ and is described by a Dirac δ function. In this case, only two other waves defined by $\mathbf{k} = \mathbf{k}_2$ and $\mathbf{k} = \mathbf{k}_3$, such that $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$ can take energy from the unstable one and eventually saturate its growth. It is the so called fixed phase approximation. We can then wite:

$$E(\mathbf{k}) = (2\pi)^3 \sum_{i=1}^{3} E_i \, \delta(\mathbf{k} - \mathbf{k}_i) \qquad (10)$$

Using this expression in equation (7) we get three coupled evolution equations:

$$\begin{aligned} (d/dt + \gamma_1)\, E_1 &= -H\, E_2 E_3\, e^{i\Omega t} \\ (d/dt + \gamma_2)\, E_2^* &= -H'\, E_1^* E_3\, e^{-i\Omega t} \qquad (11) \\ (d/dt + \gamma_3)\, E_3^* &= -H''\, E_1^* E_2\, e^{-i\Omega t} \end{aligned}$$

where $H = H(\mathbf{k}_1, \mathbf{k}_2)$, $H' = H(-\mathbf{k}_2, \mathbf{k}_3)$, $H'' = H(-\mathbf{k}_3, \mathbf{k}_2)$ and $\Omega = \omega(k_1) - \omega(k_2) - \omega(k_3)$. Using normalized amplitudes for the wave fields:

$$C_j = 1/2 \, (\partial D / \partial \omega)_{k_j}^{1/2} \, E_j \tag{12}$$

and defining real amplitudes a_j and phases ϕ_j such that

$$C_j = a_j \, e^{i \phi_j} \tag{13}$$

we can give to the system of equations (11) a symmetric form:

$$\begin{aligned} (d/dt + \gamma_1)\, a_1 &= -V a_2 a_3 \sin\Theta \\ (d/dt + \gamma_2)\, a_2 &= +V a_1 a_3 \sin\Theta \\ (d/dt + \gamma_3)\, a_3 &= +V a_1 a_2 \sin\Theta \end{aligned} \tag{14}$$

where the evolution of the phase mismatch $\Theta = \phi_1 - \phi_2 - \phi_3$ is described by the equation:

$$\frac{d\Theta}{dt} = \Omega + V \left(\frac{a_1 a_3}{a_2} + \frac{a_1 a_2}{a_3} - \frac{a_2 a_3}{a_1} \right) \cos\Theta \tag{15}$$

and the nonlinear coupling coefficient V is given by

$$V = 2H \left[\left(\frac{\partial D}{\partial \omega} \right)_{k_1} \left(\frac{\partial D}{\partial \omega} \right)_{k_2} \left(\frac{\partial D}{\partial \omega} \right)_{k_3} \right]^{-1/2} \tag{16}$$

Consider now the particular case of an unstable mode $\omega_1 = \omega(\mathbf{k}_1)$, $\mathbf{k}_1$ which saturates by subharmonic generation. In this case we have $\omega_2 = \omega_3 \simeq \omega_1/2$. And we also have $a_2 = a_3$ and $\gamma_2 = \gamma_3 > 0$. Equations (14) and (15) then reduce to:

$$\begin{aligned} (d/dt + \gamma_1)\, a_1 &= -V a_2^2 \sin\Theta \\ (d/dt + \gamma_2)\, a_2 &= V a_1 a_2 \sin\Theta \\ d\Theta/dt &= \Omega + V a_1 \, [2 - (a_2/a_1)^2] \cos\Theta \end{aligned} \tag{17}$$

It has already been shown that such a system contains chaos [1], [2]. This quite remarkable result however has a limited range of application. In the first place, the unstable spectrum always has a finite bandwidth Δ and the ideal situation of a monochromatic wave ($\Delta \rightarrow 0$) never occurs. In the second place, the two linearly stable waves a_2 and a_3 are, in general, different waves, as in the Brillouin or in the Raman scattering processes. We can avoid the first difficulty if we add to the evolution equations (11) a random term R, which phenomenologically describes the contribution of the spectrum components not entering in (10), say, the neglected nonlinear term

$$\int_{\mathbf{k} \neq \mathbf{k}_2} H(\mathbf{k}, \mathbf{k}') \; E(\mathbf{k}') \; E(\mathbf{k}-\mathbf{k}') \; e^{i\Omega t} \frac{d\mathbf{k}'}{(2\pi)^3}$$

The influence of this random term on the transition to chaos and its justification will be discussed elsewhere. Here we only want to stress that if the phases are assumed to be random the three equations (17) reduce to two and chaos is no longer observed. This quite obvious conclusion will be stated more clearly in the next Section. The second difficulty noted above can also be avoided if we keep the four equations (14) and (15). In the next Section we show that the two great limitations of the fixed phase approximation can be avoided if we study the nonlinear wave saturation using the random phase approximation.

4 — RANDOM PHASE APPROXIMATION

Let us return to equation (7). If we multiply this equation by $E^*(\mathbf{k})$ and its complex conjugate by $E(\mathbf{k})$ and sum the results, we get:

$$(2\pi)^3 \; (d/dt + 2\gamma_k) \; |E(\mathbf{k})|^2$$
$$= -2 \int \mathrm{Re} \; [H(\mathbf{k}, \mathbf{k}') \; E^*(\mathbf{k}) \; E(\mathbf{k}') \; E(\mathbf{k}'') \; e^{i\Omega t}] \; d\mathbf{k}' \tag{18}$$

If we write the field amplitude as $E(\mathbf{k}) = |E(\mathbf{k})| \exp(i\,\phi_k)$ and if we assume that the phase behaves randomly we can make a statistical average of equation (18) over the phase. It is easy to see that a statistical average over a random phase leads to $< E(\mathbf{k}) > = 0$ and $< E^*(\mathbf{k})\, E(\mathbf{k}') > = |E(\mathbf{k})|^2\, \delta(\mathbf{k}-\mathbf{k}')$. In order to calculate the mean of product $E^*(\mathbf{k})\, E(\mathbf{k}')\, E(\mathbf{k}'')$ entering in equation (18) we take an approximate solution of equation (7), assuming that $d/dt \simeq i\Delta$ where $\Delta >> \gamma_k$ is the wave frequency width. We get then:

$$i\Delta\,(2\pi)^3\, E(\mathbf{k}) \simeq -\int H(\mathbf{k},\mathbf{k}')\, E(\mathbf{k}')\, E(\mathbf{k}'')\, e^{i\Omega t}\, d\mathbf{k}' \qquad (19)$$

Using this result we can write:

$$\begin{aligned} & i\Delta(2\pi)^3 < E^*(\mathbf{k})\, E(\mathbf{k}')\, E(\mathbf{k}'') > \\ \simeq & \int H^*(\mathbf{k},\mathbf{s}) < E^*(\mathbf{s}')\, E^*(\mathbf{k}-\mathbf{s}')\, E(\mathbf{k}')\, E(\mathbf{k}-\mathbf{k}') > e^{-i\Omega' t}\, d\mathbf{s}' \\ & - \int H(\mathbf{k}',\mathbf{s}'') < E^*(\mathbf{k})\, E(\mathbf{s}'')\, E(\mathbf{k}'-\mathbf{s}'')\, E(\mathbf{k}-\mathbf{k}') > e^{-i\Omega'' t}\, d\mathbf{s}'' \qquad (20) \\ & - \int H(\mathbf{k}-\mathbf{k}',\mathbf{s}''') < E^*(\mathbf{k})\, E(\mathbf{k}')\, E(\mathbf{s}''')\, E(\mathbf{k}-\mathbf{k}'-\mathbf{s}''') > e^{-i\Omega''' t}\, d\mathbf{s}''' \end{aligned}$$

where $\Omega' = \Omega(\mathbf{k}' \to \mathbf{s}',\ \mathbf{k}'' \to \mathbf{k}-\mathbf{s}')$ and Ω'' and Ω''' are defined in a similar way. Now, in the random phase approximation it is easy to see that:

$$\begin{aligned} & < E^*(\mathbf{s}')\, E^*(\mathbf{k}-\mathbf{s}')\, E(\mathbf{k}')\, E(\mathbf{k}-\mathbf{k}') > \\ & \quad \simeq < E^*(\mathbf{s}')\, E(\mathbf{k}') > < E^*(\mathbf{k}-\mathbf{s}')\, E(\mathbf{k}-\mathbf{k}') > \qquad (21) \\ & \quad \simeq |E(\mathbf{s}')|^2\, |E(\mathbf{k}-\mathbf{k}')|^2\, \delta(\mathbf{s}'-\mathbf{k}') \end{aligned}$$

Using (20) and (21) we can write the statistical average of equation (18) in the form.

$$\begin{aligned} & (2\pi)^3\, (d/dt + 2\gamma_k)\, N_k \\ & \quad = \int w(\mathbf{k},\mathbf{k}')\, [N_{k'} N_{k''} - N_k N_{k''} - N_k N_{k'}]\, d\mathbf{k}' \end{aligned} \qquad (22)$$

where $N_k = C_k^2$ are the number of photons of wavevector **k** and C_k is defined by equation (12). The nonlinear coupling coefficient $w(\mathbf{k}, \mathbf{k}')$ is now given by:

$$w(\mathbf{k}, \mathbf{k}') = -(4\pi/\varepsilon_0) \frac{|H(\mathbf{k}, \mathbf{k}')|^2 \; \delta(\Omega)}{\left(\frac{\partial D}{\partial \omega}\right)_k \left(\frac{\partial D}{\partial \omega}\right)_{k'} \left(\frac{\partial D}{\partial \omega}\right)_{k''}} \tag{23}$$

Assuming further that the main components of the wave spectrum lie in the region of wavevectors around $\mathbf{k} = \mathbf{k}_i$ (with $i = 1, 2, 3$) we can use the properties of the δ function $\delta(\Delta)$ and get, after integration in $\mathbf{k}'$,

$$(d/dt + \gamma_1)\; N_1 = a_1 (N_2N_3 - N_1N_3 - N_1N_2) \tag{24}$$

where $N_i = N_{k_i}$ (for $i = 1,2,3$), $\gamma_1 = 2\,\gamma_{k_1}$ and

$$a_1 = w(\mathbf{k}, \mathbf{k}')\; [2\pi (\partial\Omega / \partial \mathbf{k}')_{k_2}]^{-1} \tag{25}$$

For positive energy waves, we always have $a_1 > 0$ [3]. Coupled with this evolution equation for N_1 we have two similar equations for N_2 and N_3. If we normalize the time with respect to the growth rate of the unstable mode N_1, with the aid of a new time variable $\tau = |\gamma_1|\, t$, it is easily seen that we can write the three coupled equations in the form:

$$\begin{aligned} dN_1 / d\tau &= N_1 + A_1 (N_2N_3 - N_1N_3 - N_1N_2) \\ dN_{2,3} / d\tau &= -\Gamma_{2,3} N_{2,3} - A_{2,3} (N_2N_3 - N_1N_3 - N_1N_2) \end{aligned} \tag{26}$$

where $A_i = a_i / |\gamma_1| > 0$ and $\Gamma_{2,3} = \gamma_{2,3} / |\gamma_1| > 0$ are the linear damping rates of the stable modes N_2 and N_3. These equations are quite similar to the Lotka-Volterra equations for three populations in competition, but we will see that the behaviour of their solution is quite different.

Equations (26) have two singular points defined by:

$$\begin{aligned} &P_0: \quad N_1 = N_2 = N_3 = 0 \\ &P_1: \quad N_1 = A_2\Gamma_2\Gamma_3 / B, \quad N_2 = A_2\Gamma_3 / B, \quad N_3 = A_3\Gamma_2 / B \end{aligned} \tag{27}$$

where

$$B = A_1 (A_2\Gamma_3 + A_3\Gamma_2) - A_2A_3 \tag{28}$$

The origin P_0 is always unstable and is of the node-saddle type. Its stable manifold is the plane $P_0N_2N_3$ and the unstable manifold is the axis P_0N_1. The second singular point P_1 exists in the first octant of the phase space only if $B > 0$. Equation (26) has a physical meaning only when the point representing the system in phase space belongs to the first octant, because the numbers of photons are positive quantities, $N_i > 0$. Then, in order to get an instability saturation the singular point P_1 must be accessible to the system. The necessary condition for the instability saturation is then $B > 0$.

Furthermore, if we use the Hurwitz criterion we can easily show that P_1 is always stable and shows a focus behaviour, provided that $B > 0$. We can then say that the configuration in phase space is always the same and no bifurcation parameter can be defined. The conclusion is that, unlike the three dimensional Lotka-Volterra equations, no chaos can be generated in this system, because it would have to exist even for very small values of N_i when the nonlinear terms are negligible.

These results are confirmed by the numerical integration of equations (26). When $B = 0$ the unstable mode boes to infinity, as well as the linearly stable modes. When $B > 0$ the unstable mode attains a maximum at a time τ_c and then slowly decays to the saturation level, as shown in Fig. 1. This maximum of N_1 grows with Γ, as well as the saturation time τ_c. For high enough values of Γ a series of oscillations around P_1 are observed with decreasing amplitudes (Fig. 2), revealing the focus nature of P_1. The behaviour is essentially of the same nature for $\Gamma_2 \neq \Gamma_3$.

5 — CONCLUSIONS

We have studied the instability saturation by nonlinear mode coupling and we have discussed the behaviour of the saturation wave amplitude levels in two extreme approximations. The first one corresponds to a nearly monochromatic unstable spectrum and it is the fixed phase approximation ($\Delta \tau_c << 1$) . In this case the

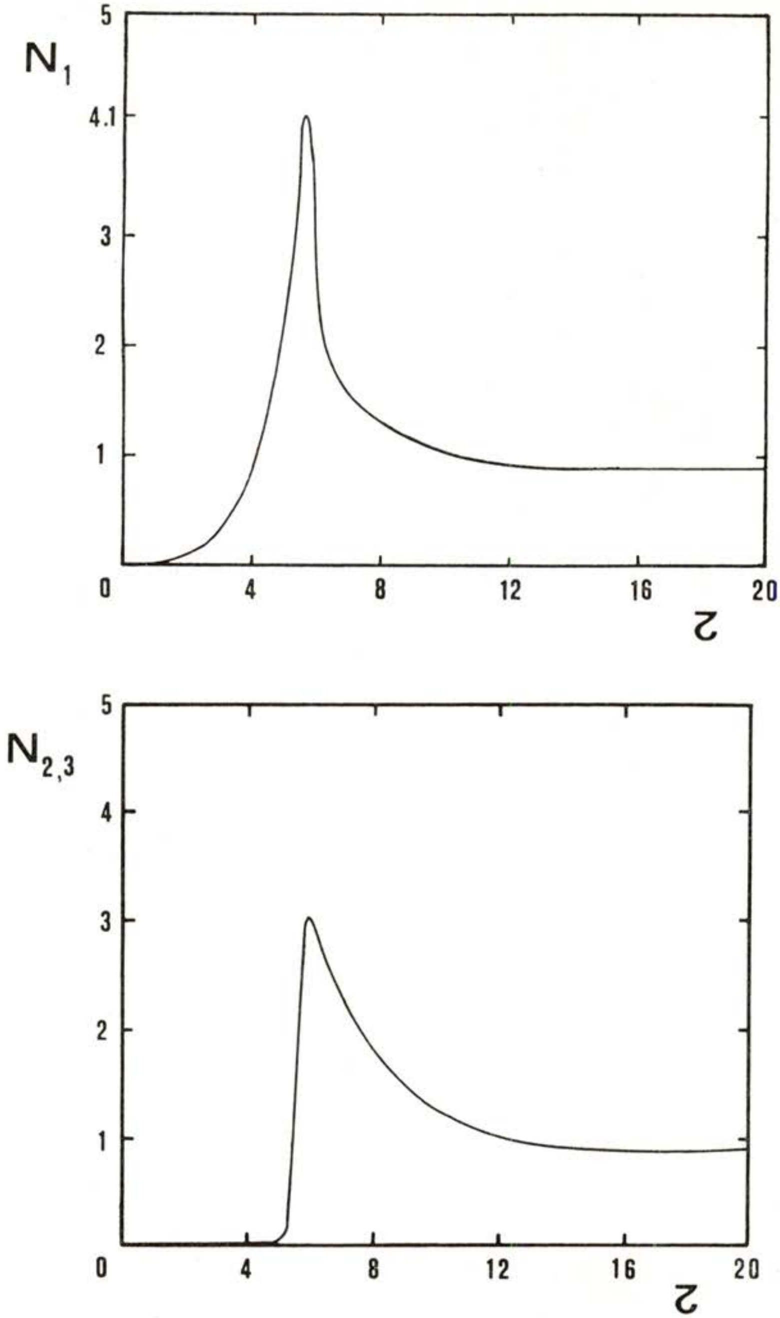

Fig. 1 — Instability saturation in the random phase approximation, for $A_1 = A_2 = A_3 = 1$ and $\Gamma_2 = \Gamma_3 = 1$. (a) Intensity of the first mode; (b) Intensity of the second and third modes (arbitrary units).

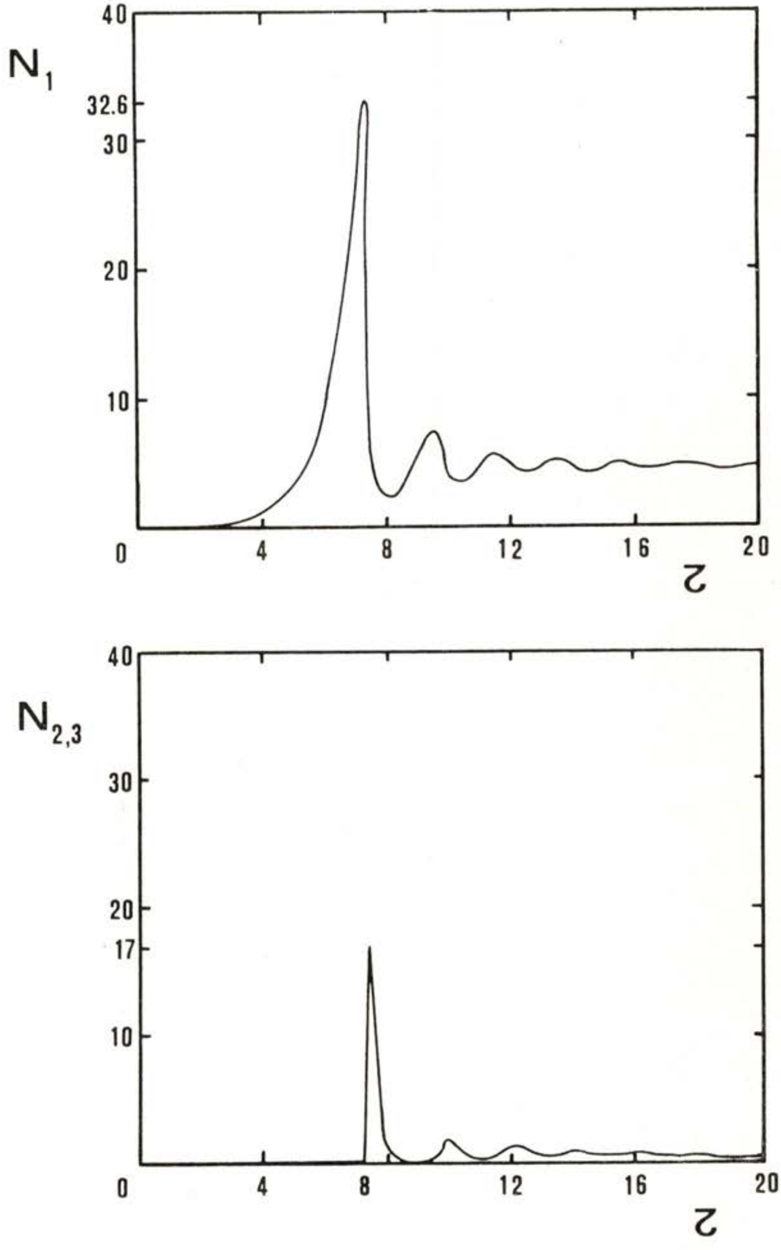

Fig. 2 — The same as in Fig. 1, but for $\Gamma_2 = \Gamma_3 = 100$.

wave amplitudes starting from zero can asymptotically attain a limit cycle regime which eventually bifurcates to chaos. This was shown by numerical calculations, assuming that the two stable waves are equal ($k_2 = k_3$, $a_2 = a_3$) , which means that we have sub-harmonic generation ($\omega_2 = \omega_3 \simeq 2\omega_0$) . Of course, if chaos is attained, the unstable spectrum broadens and the waves can no more be considered as monochromatic.

The second extreme case is that of a broad-band unstable spectrum, described by the random phase approximation ($\Delta \tau_c >> 1$). Making no restriction on the character of the stable waves ($k_2 \neq k_3$, $N_2 \neq N_3$) we have shown that in this case the system tends to a stable focus and no bifurcation to chaos is observed.

It is our intuition that the random phase approximation is more appropriate to describe the physical phenomena because, even if the unstable spectrum is nearly monochromatic it becomes broader when the chaotic behaviour is observed. But only a detailed study of the intermediate case described by the fixed phase equations with a noise term can eventually give a qualified answer to this intuition.

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