NUCLEAR HYDRODYNAMICS (*)

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ABSTRACT — Based on the local equilibrium assumption and taking as wave function a Slater determinant, the equations of motion and boundary conditions for the first sound are obtained from a variational derivation based on the quantum mechanical lagrangian. Assuming density dependent δ forces, it is shown that in the classical limit the equilibrium density is $\rho_0(r) = \rho_0(0)\,\Theta(R\text{-}r)$, where $\rho_0(0)$ is the nuclear matter equilibrium density.

1 — INTRODUCTION

Giant resonances in atomic nuclei are highly excited states in which an appreciable fraction of the nucleons of a nucleus move in a coherent manner.

On the microscopic level the random phase approximation provides a very detailed description of collective vibrations. It requires, however, a considerable numerical effort, which might obscure the simple physical relations pertinent to strongly collective excitations. Fluid dynamical methods in application to giant multipole resonances [2-11] aim at understanding salient features of these collective modes, without entering into the complexity of detailed numerical descriptions.

In order to reach a deeper understanding of the physical processes associated with the behaviour of atomic nuclei, it is desirable to separate detailed aspects of nuclear properties, which often appear due to shell effects, from gross properties depending

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smoothly on the mass number A. This suggests, therefore, an explanation on the basis of fluid dynamical approximations, which may be formulated in terms of such quantities as matter and current densities, denoted respectively by ρ and $\mathbf{j}$, pressure tensor P_{ij}, etc.

In this note, we will restrict the discussion to hydrodynamics [1-3], which is the simplest example of such an approximation. In the hydrodynamical case, the main assumption is the use of the Thomas-Fermi approximation or, equivalently, that the sphericity of the Fermi surface in momentum space is preserved during the nuclear motion.

Our purpose is to derive the macroscopic equations of motion, which characterize first sound, starting from a microscopic basis. As wave function we consider the following Slater determinant

$$|\phi> = \exp(i\hat{Q}\hbar^{-1})|\phi_f> \quad , \tag{1}$$

where $|\phi_f>$ is, among the Slater determinants leading to the density ρ_f the one which minimizes the expectation value of the energy. Therefore the distribution function, associated to $|\phi_f>$, may be written as follows

$$f_f = \Theta(p_f^2(\mathbf{r}) - p^2) \quad , \tag{2}$$

assuming the value 1 when $p_f^2(\mathbf{r}) > p^2$ and zero otherwise. In this way the Pauli principle is obviously taken into account.

In order to have an appropriate description of the time evolution of the system, we must allow the distribution function to acquire time odd components, which is done with the help of the time even generator $\hat{Q}$. In this note $\hat{Q}$ is just a local field

$$\hat{Q} = \sum_{i=1}^{A} \chi(\mathbf{r}_i, t) \quad . \tag{3}$$

Since we are interested in the classical limit of nuclear dynamics, we restrict our discussion to the leading orders of appropriate Wigner-Kirkwood expansions. To avoid cumbersome notations, we find it most often convenient to denote by the same symbol an operator and its Wigner transform. The density matrix $\hat{\rho}$ is the only exception. In this case, we denote the distribution by $f(\mathbf{r}, \mathbf{p}, t)$, in order to avoid confusion with the density $\rho(\mathbf{r}, t)$.

2 — THE STATIC PROBLEM

The equilibrium distribution function f_0 is obtained by minimizing the energy density functional $W[f]$

$$W[f] = \int d\Gamma_1 f(1)\, p_1^2/(2m) + (1/2!) \iint d\Gamma_1 d\Gamma_2 f(1) f(2) v_{12} + (1/3!) \iiint d\Gamma_1 d\Gamma_2 d\Gamma_3 f(1) f(2) f(3) v_{123} + \dots \tag{4}$$

and by taking into account the subsidiary condition

$$A = \int d\Gamma\ f \quad , \tag{5}$$

where the quantities v_{12}, v_{123}, ..., stand, respectively, for the two-body, three-body, ... interactions. A is the particle number and $d\Gamma$ is given by the following expression

$$d\Gamma = g\ d^3r\ d^3p\ (2\pi\hbar)^{-3} \ . \tag{6}$$

The distribution function describing a system instantaneously at rest is given by (2). Since the only quantity on which f_f depends is $p_f^2(\mathbf{r})$, it is clear that $W[f_f]$ may be written as a functional of the density ρ_f associated to f_f,

$$\rho_f = g \int d^3p\ (2\pi\hbar)^{-3}\ f_f \quad , \tag{7}$$

$$W[f_f] = E[\rho_f] = \int_D d^3r\ F(\rho_f) \quad , \tag{8}$$

where the domain D is the region where $p_f^2(\mathbf{r})$ is positive.

A simplified hamiltonian with two-body and three-body δ forces is considered,

$$v_{12} = t_0\ \delta(\mathbf{r}_1 - \mathbf{r}_2) \quad , \tag{9}$$

$$v_{123} = t_3\ \delta(\mathbf{r}_1 - \mathbf{r}_2)\ \delta(\mathbf{r}_2 - \mathbf{r}_3) \quad . \tag{10}$$

For such an hamiltonian $F(\rho_f)$ is

$$F(\rho_f) = (3/10m)\,\rho_f\, p_f^2 + a_2\,\rho_f^2 + a_3\,\rho_f^3 \tag{11}$$

We choose the following Skyrme parameters: $a_2 = -408.4$ MeV fm^3 and $a_3 = 1079.4$ MeV fm^6. This choice is made in order to enable comparision with the results obtained in ref. [3] for a calculation of first sound in finite droplets of nuclear matter with smooth surface and therefore related to an energy functional which includes, besides the volume terms appearing in (11), also other terms involving derivatives of the density.

We now proceed to a general variation of the energy functional E taking into account as a subsidiary condition that the particle number A remains constant

$$\begin{aligned}\delta(E-\lambda_0 A) &= \int_D d^3r\ \delta\rho\ (dF/d\rho - \lambda_0)\\ &\quad + \int_\Sigma d\Sigma\ (\delta\mathbf{R}\cdot\mathbf{n})\ (F(\rho) - \lambda_0\,\rho)\,.\end{aligned} \tag{12}$$

$\delta\mathbf{R}$ denotes the displacement of the boundary Σ of the domain D and $\mathbf{n}$ is the outwards normal. The equilibrium density ρ_0 is the solution of the following set of equations

$$(dF/d\rho)_{\rho=\rho_0} = \lambda_0 \quad , \tag{13}$$

$$(F(\rho_0) - \lambda_0\,\rho_0)_{r=R} = 0 \quad , \tag{14}$$

where R is the radius of the spherical nucleus.

Equation (13) implies that ρ_0 is independent of $\mathbf{r}$. Combining (13) and (14) it follows that the value of ρ_0 is obtained by minimizing the total energy $A\,F(\rho)/\rho$,

$$(d[\rho^{-1}F(\rho)]/d\rho)_{\rho=\rho_0} = 0\,. \tag{15}$$

This means that ρ_0 is the equilibrium density of nuclear matter. From now on we will be considering the equilibrium density

$$\rho_0(\mathbf{r}) = \rho_0(0)\ \Theta(R-r) \quad , \tag{16}$$

where the radius R is fixed by $\rho_0(0)$ and A.

3 — TIME EVOLUTION

The distribution function corresponding to the Slater determinant (1) is

$$f = f_f + \{ f_f , \chi \} + (1/2) \{ \chi , \{ \chi , f_f \} \} + \ldots \tag{17}$$

Assuming that the field χ is small, we have that the density and the current are

$$\rho = g \int d^3p \ (2 \pi \hbar)^{-3} \ f \simeq \rho_f \quad , \tag{18}$$

$$\mathbf{j} = g \int d^3p \, (2 \pi \hbar)^{-3} \ f \ \mathbf{p}/m \simeq (\rho_f/m) \ \nabla \chi . \tag{19}$$

From the quantum mechanical lagrangian

$$L = i \hbar < \phi | \dot{\phi} > - < \phi | H | \phi > \quad , \tag{20}$$

we obtain in the classical limit the following lagrangian for the fields χ and ρ_f

$$L = \int_D d^3r \ \{ -\dot{\chi} \rho_f - (\rho_f/2m) \ (\nabla \chi)^2 - F(\rho_f) \} \quad , \tag{21}$$

where

$$< \phi_f | H | \phi_f > = \int d^3r \ F(\rho_f) \quad . \tag{22}$$

When we minimize the action integral, we take into account the conservation of the particle number by introducing an appropriate Lagrange multiplier λ,

$$\begin{aligned} \delta \int dt \, (L + \lambda A) = & \int dt \, \{ \int_D d^3r \ \delta \chi \ [\dot{\rho}_f + (1/m) \ \nabla \cdot (\rho_f \nabla \chi)] \\ & + \int_D d^3r \ \delta \rho_f [- \dot{\chi} - (1/2m) \ (\nabla \chi)^2 - dF/d \rho_f + \lambda] \\ & + \int_\Sigma d\Sigma \ \delta \chi \ \rho_f \ \hat{\mathbf{n}} \cdot (\dot{\mathbf{R}} - (1/m) \nabla \chi) \\ & + \int_\Sigma d\Sigma \ (\delta \mathbf{R} \cdot \hat{\mathbf{n}}) \ [- \dot{\chi} \rho_f - (\rho_f/2m) \ (\nabla \chi)^2 - F(\rho_f) + \lambda \rho_f] \ . \end{aligned} \tag{23}$$

By considering arbitrary variations of the fields χ and ρ_f, the following equations of motion are obtained

$$\dot{\rho}_f + (1/m)\,\nabla \cdot (\rho_f \nabla \chi) = 0 \quad , \tag{24}$$

$$\dot{\chi} + (1/2m)\,(\nabla \chi)^2 + dF/d\rho_f - \lambda = 0 \quad . \tag{25}$$

Equation (24) is obviously the continuity equation and equation (25) leads to the 'Euler type' equation

$$\partial_t\, \mathbf{j} = -(\rho_f/m)\,\nabla\,(dF/d\rho_f) \quad . \tag{26}$$

The two following boundary conditions are obtained

$$\dot{\chi}\,\rho_f + (\rho_f/2m)\,(\nabla \chi)^2 + F(\rho_f) - \lambda\,\rho_f \,|_{\,r=R} = 0\,, \tag{27}$$

$$\rho_f\,(\dot{\mathbf{R}} - (1/m)\,\nabla \chi)\cdot \hat{\mathbf{n}}\,|_{\,r=R} = 0\ . \tag{28}$$

The equations (25) and (27) imply the boundary condition (15) at the surface. This means that at the surface ρ_f is equal to ρ_0 and therefore we recover the well known first sound [1-3] boundary condition

$$\rho_f^{(1)}\,|_{\,r=R} = 0 \quad , \tag{29}$$

where $\rho_f^{(1)} = \rho_f - \rho_0$.

From equations (24) and (25) we obtain the first sound equation for ρ_f

$$\ddot{\rho}_f = (1/m)\,\nabla \cdot (\rho_f \nabla (dF/d\rho_f)) \quad . \tag{30}$$

If we linearize this equation we obtain in the interior of the nucleus

$$-\omega^2\,\rho_f^{(1)} = c_f^2\,\Delta\,\rho_f^{(1)} \quad , \tag{31}$$

with the first sound velocity

$$c_f = (p_F/m)\,\sqrt{(1 + F_0)/3} \quad , \tag{32}$$

and where the Landau parameter F_0 is

$$F_0 = (3m/p_F^2)\,\sum_\sigma a_\sigma\,\sigma(\sigma - 1)\,\rho_0^{\sigma-1}. \tag{33}$$

The solutions $\rho_f^{(1)}$ have the following analytical form inside the nucleus

$$\rho_f^{(1)} \propto j_l(kr)\ Y_{lm} \quad , \tag{34}$$

where j_l is a spherical Bessel function and $k = \omega / c_f$.

The energies of the first compressive mode according to the present formalism are shown, for different values of l, in the following table, for a nucleus with $A = 208$ and compared with the corresponding energies obtained by solving eq. (30) for a nucleus with a smooth surface [3] based on a more sophisticated formalism, allowing for quantum corrections through the inclusion of the so called surface terms.

TABLE — First sound eigenfrequencies (in MeV) for the first compressional mode for a nucleus with $A = 208$. The energies in the first line are taken from ref. [3], those in the second line are obtained according to the square well model density.

$l=0$	$l=1$	$l=2$	$l=3$
18.4	25.3	30.9	35.5
18.5	26.4	33.9	41.1

4 — CONCLUSION

In this note we have derived the first sound equations of motion and respective boundary conditions, starting from a microscopic point of view, where determinants are taken as trial wave functions and local equilibrium is assumed.

Actually the local equilibrium assumption is not realistic for atomic nuclei at very low temperatures, because then the mean free path λ of the nucleons in nuclei is of the order of the typical wavelength R (nuclear radius) and therefore the basic physical condition for first sound modes (namely $\lambda \ll R$) is generally not met. It is well known that nuclear giant resonances may be obtained in a fluid dynamical picture by means of the generalized scaling approach [3-10]. However in order to obtain a description of low lying modes in a fluid dynamical approach, one has to go beyond the generalized scaling approach. One possible way of obtaining low lying modes is to allow for the interplay between first sound and the generalized scaling approach [2, 11].

APPENDIX

In eq. (23) we had to perform a partial integration with respect to time leading to a surface term and to a volume term. In order to understand how the surface contribution appears, we consider the integral $\int_{t_1}^{t_2} dt \int_D d^3r\ G(\mathbf{r}, t)$ such that

$$\delta \int_D d^3r\, G(\mathbf{r}, t_1) = \delta \int_D d^3r\, G(\mathbf{r}, t_2) = 0 \tag{A.1}$$

Then, we have

$$\int_{t_1}^{t_2} dt \frac{d}{dt} \int_D d^3r\, G(\mathbf{r}, t) = \int_{t_1}^{t_2} \left\{ \int_D d^3r\, \partial_t G + \int_\Sigma d\Sigma\, (\dot{\mathbf{R}} \cdot \hat{\mathbf{n}})\, G(\mathbf{r}, t) \right\} = 0 \tag{A.2}$$

and, in particular, if $G = \delta\chi\, \rho$, we will have

$$\int_{t_1}^{t_2} dt \left\{ \int_D d^3r\, (\delta\dot{\chi}\, \rho + \delta\chi\, \dot{\rho}) + \int_\Sigma d\Sigma\, (\dot{\mathbf{R}} \cdot \hat{\mathbf{n}})\, \delta\chi\, \rho \right\} = 0 \tag{A.3}$$

so that

$$\int_{t_1}^{t_2} dt \left\{ \int_D d^3r\, \delta\dot{\chi}\, \rho \right\} = \int_{t_1}^{t_2} dt \left\{ - \int_D d^3r\, \delta\chi\, \dot{\rho} + \int_\Sigma d\Sigma\, (\dot{\mathbf{R}} \cdot \hat{\mathbf{n}})\, \delta\chi\, \rho \right\} \tag{A.4}$$

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