ON THE QUANTUM HARMONIC OSCILLATOR WITH TIME-DEPENDENT FREQUENCY

FRANCISCO M. FERNÁNDEZ AND EDUARDO A. CASTRO

Prrograma QUINOR, Facultad de Ciencias Exactas, UNLP, Calle 47 y 115, C.C. 962, 1900 La Plata, Argentina.

ABSTRACT-The time evolution operator for the quantum harmonic oscillator with time-dependent frequency is exactly obtained as a product of unitary operators. The calculation is greatly simplified by taking into account the equations of motion for the coordinate and momemtum operators in the Heisenberg representation.

1.INTRODUCTION

The driven harmonic oscillator is a useful model for many physical systems of practical interest. Among them we mention lattice vibrations in solids and energy transfer in molecular collisions [1-3]. It is also suitable for teaching purposes because the Schrödinger equation is exactly solvable [1,4].

In this paper we are interested in the harmonic oscillator with time-dependent frequency (units are used so that $\hbar = m = 1$).

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}w^2(t)q^2 \qquad (1)$$

where $[q,p] = i$. It is our purpose to obtain the time evolution operator $U(t)$

that satisfies the equation

$$i\frac{d}{dt}U = \mathcal{H}\,U \qquad (2)$$

with the boundary condition $U(0)=1$. The solution of Eq. (2) can be written

$$U=U_1U_2U_3, \quad U_j = \exp(-ia_j(t)A_j) \qquad (3)$$

where $a_j(0)=0$ and

$$A_1 = \frac{1}{2}q^2, \quad A_2 = \frac{1}{2}(qp+pq), \quad A_3 = \frac{1}{2}p^2 \qquad (4)$$

because the operators A_j form a Lie algebra:

$$[A_1, A_2] = 2i\,A_1, \quad [A_1, A_3] = i\,A_2,$$
$$[A_2, A_3] = 2i\,A_3 \qquad (5)$$

On introducing Eq. (3) into Eq. (2) we are led to a set of nonlinear differential equations that determine exactly the functions $a_j(t)$ [5]. Besides, nonlinear functions of the a_j's can be found that satisfy classical equations of motion [3,6].

It is shown in the present paper that the latter functions and their relationship with the former ones can be much more easily obtained from the equations of motion for the coordinate and momentum operators in the Heisenberg representation. The procedure is discussed in Sec. 2 and its generalization to more complex problems is briefly outlined in Sec. 3.

2. THE METHOD

In the Heisenberg representation any linear operator O is written

$$O_t = U^\dagger O\, U \qquad (6)$$

which satisfies the well known quantum mechanical equation of motion

$$\frac{d}{dt} O_t = U^\dagger \frac{\partial O}{\partial t} U + i\, U^\dagger [\mathcal{H}, O_t] U \qquad (7)$$

In particular, for the coordinate and momentum we have

$$\frac{d}{dt} q_t = p_t \, .$$

$$\frac{d}{dt} p_t = - w^2 q_t \qquad (8)$$

that lead to

$$\frac{d^2}{dt^2} q_t = - w^2 q_t \qquad (9)$$

Since the coordinate and momentum form a complete set of observables for the oscillator we expect to obtain all the relevant dynamical information about the system from Eqs. (8).

It is clear that the solution of Eqs. (8) must be of the form

$$q_t = Q_1(t)\, p + Q_2(t)\, q$$
$$p_t = P_1(t)\, p + P_2(t)\, q \qquad (10)$$

Where $Q_1(0)=P_2(0)=0$ and $Q_2(0)=P_1(0)=1$. On introducing (10) into (8) we obtain the classical equations of motion

$$\dot{Q}_j = P_j \, , \quad \dot{P}_j = - w^2\, Q_j, \quad j=1,2 \qquad (11)$$

where the dot means time derivative, or

$$\ddot{Q}_j = - w^2\, Q_j, \quad j=1,2 \qquad (12)$$

Only three of the four functions Q_j and P_j are independent since $[q_t, p_t] = U^\dagger [q, p] U = i$. Therefore,

$$P_1 Q_2 - P_2 Q_1 = 1 \qquad (13)$$

which is actually the Wronskian for the

two independent solutions of (12).

In order to obtain the relationship between the functions P_j, Q_j and the functions a_k we make use of the results in the Appendix. A straightforward calculation shows that

$$U_1^\dagger p\, U_1 = p - a_1 q,$$
$$U_2^\dagger p\, U_2 = \exp(-a_2)p.$$
$$U_2^\dagger q\, U_2 = \exp(a_2)q.$$
$$U_3^\dagger q\, U_3 = q + a_3 p. \tag{14}$$

which lead to (cf Eq. (10))

$$P_1 = \exp(-a_2) - a_1 a_3 \exp(a_3),$$
$$P_2 = -a_1 \exp(a_2),$$
$$Q_1 = a_3 \exp(a_2),$$
$$Q_2 = \exp(a_2), \tag{15}$$

Finally, the time evolution operator can be written

$$U= \exp(\frac{iP_2}{2Q_2}q^2)\exp(-\frac{i}{2}\ln Q_2(qp+pq))\exp(-\frac{iQ_1}{2Q_2}p^2)$$

$$\tag{16}$$

which is the result obtained by Pechukas and Light [6] through a rather more involved procedure. It is worth noticing that the time dependence of the wave-function $\psi(t)=U(t)\psi(0)$ is determined by the solutions of the equations of motion for the classical analog of (1).

3. CONCLUSIONS

The method developed in the present paper is much simpler than the methods reported previously [3,5,6] because a remarkably less number of commutators is required. The reader can convince himself by obtaining the differential equations for the a's from Eqs. (2) and (3) [3,5]. In addition to this, the relationship between the functions a_j and the functions P_j and Q_j follows immediately from the form of the coordinate and momentum operators in the Heisenberg representation.

It must be noticed that the phase factor in U cannot be obtained from the quantum-mechanical equations of motion. For instance, suppose that

$$\mathcal{H} = \sum_{j=1}^{5} \mathcal{H}_j(t)A_j \tag{17}$$

where $A_4 = q$ and $A_5 = p$. Such an operator occurs in the treatment of a collinear collision of an atom with a diatomic molecule [3]. In this case the time evolution operator can be written

$$U = \prod_{j=1}^{6} U_j, \tag{18}$$

where $A_6 = 1$. Clearly, the phase factor $U_6 = \exp(-ia_6)$ does not appear in q_t or p_t. However, it can be easily obtained from (2). For example, direct inspection of the terms in $(i\frac{d}{dt}U)U^\dagger = \mathcal{H}$ shows that $A_6 = -A_5 a_4$.

Owing to its great simplicity the present procedure can be applied to more complex problems. We have obtained the time evolution operator for an n-dimensional oscillator with linear and bilinear terms in coordinate and momenta and arbitrary time dependent coefficients [7]. Such a model originates, for instance, in the semiclassical approximation to the collision between two molecules [2, 3, 8]. As far as we know only approximate solutions had been tried before for the simplest cases [2, 8].

4. APPENDIX

Throughout this paper we had to calculate expressions of the form

$$\tilde{B}(a) = T^{-1}BT \qquad (A1)$$

where $T = \exp(-iaA)$ and A and B are linear operators. This can be easily done by taking into account that

$$\frac{d}{da}\tilde{B} = iT^{-1}[A,B]T \qquad (A2)$$

Therefore, the nth derivative of B with respect to a can be written

$$\tilde{B}^{(n)} = i^n T^{-1}B_nT, \qquad (A3)$$

where

$$B_n = [A,B_{n-1}], \ n = 1,2,...., \ B_0 = B \qquad (A4)$$

We are thus led to the Taylor series

$$\tilde{B} = \sum_{j=0}^{\infty} \frac{(ia)^j}{j!}B_j \qquad (A5)$$

which reduces to a polynomial of degree n when $[A,B_n] = 0$. As an example consider $A = p^2$ and $B = q$ (n=1 in this case).

In some cases it is easier to obtain $\tilde{B}$ by integrating the differential equation that comes from (A3). If, for instance, it is found that $\tilde{B}' = g\tilde{B}$ then $\tilde{B} = e^{ga} B$. This certainly happens when $A = qp+pq$ and $B = p$ or $B = q$. Other situations are treated exactly in the same way.

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